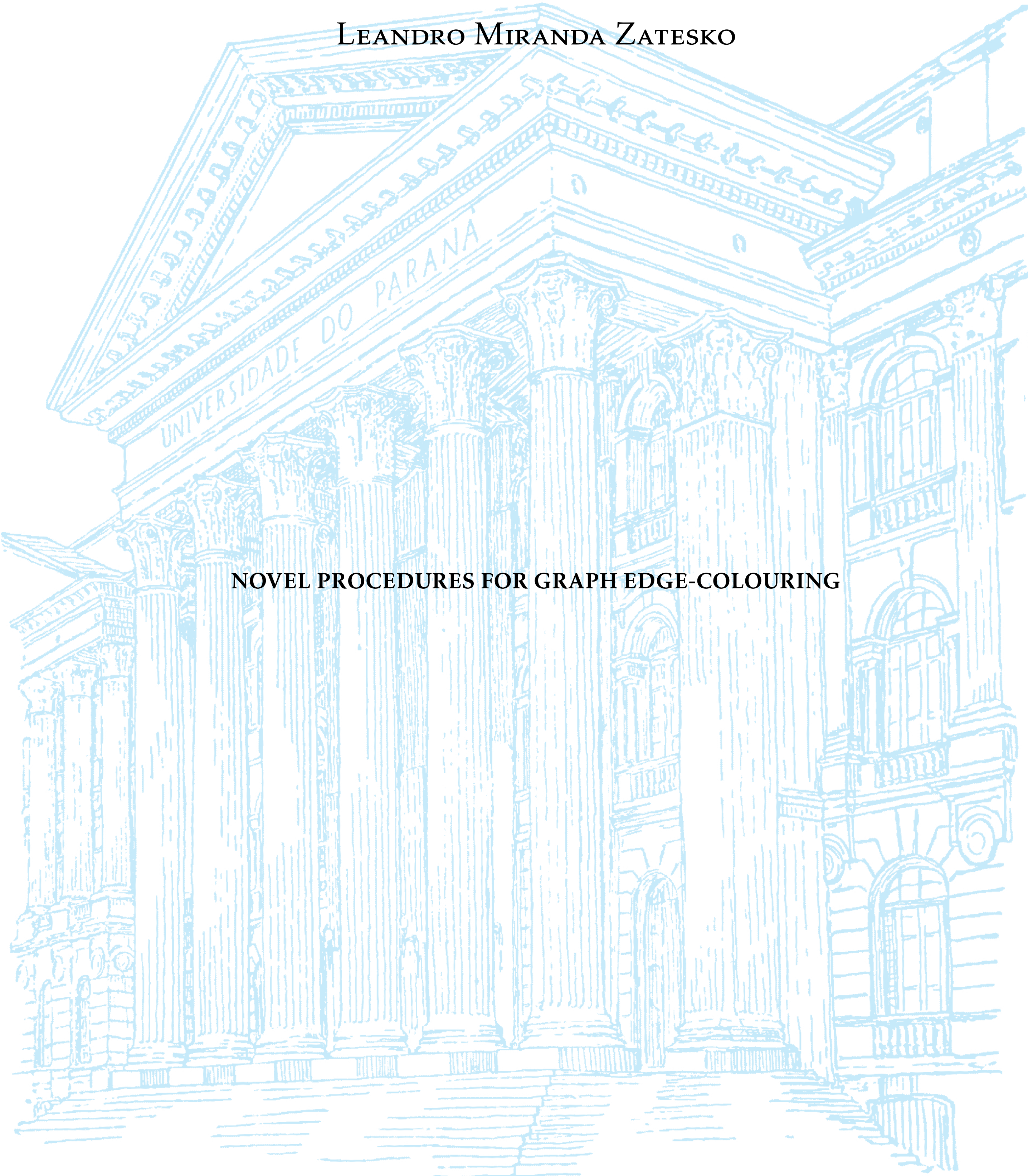
Federal University of Paraná

Leandro Miranda Zatesko



NOVEL PROCEDURES FOR GRAPH EDGE-COLOURING

Curitiba, PR, Brazil
2018

Leandro Miranda Zatesko

NOVEL PROCEDURES FOR GRAPH EDGE-COLOURING

A doctoral thesis submitted in partial fulfilment of the requirements for the award of the degree of Doctor in Computer Science, in the Graduate Programme in Informatics, Faculty of Exact Sciences, Federal University of Paraná.

Field: *Computer Science*
Advisor: Dr Renato Carmo
Co-advisor: Dr André Luiz Pires Guedes

Curitiba, PR, Brazil
2018

Catalogação na Fonte: Sistema de Bibliotecas, UFPR
Biblioteca de Ciência e Tecnologia

Z36n Zatesko, Leandro Miranda
Novel procedures for graph edge-colouring / Leandro Miranda Zatesko. – Curitiba, 2018.

Tese - Universidade Federal do Paraná, Setor de Ciências Exatas, Programa de Pós-Graduação em Informática, 2018.

Orientador: Renato Carmo. Coorientador: André Luiz Pires Guedes.

1. Teoria dos grafos. 2. Algorítmos computacionais. 3. Computação. I. Universidade Federal do Paraná. II. Carmo, Renato. III. Guedes, André Luiz Pires. IV. Título.

CDD: 005.1

Bibliotecária: Vanusa Maciel CRB- 9/1928

MINISTÉRIO DA EDUCAÇÃO
SETOR SETOR DE CIENCIAS EXATAS
UNIVERSIDADE FEDERAL DO PARANÁ
PRÓ-REITORIA DE PESQUISA E PÓS-GRADUAÇÃO
PROGRAMA DE PÓS-GRADUAÇÃO INFORMÁTICA

TERMO DE APROVAÇÃO

Os membros da Banca Examinadora designada pelo Colegiado do Programa de Pós-Graduação em INFORMÁTICA da Universidade Federal do Paraná foram convocados para realizar a arguição da tese de Doutorado de **LEANDRO MIRANDA ZATESKO** intitulada: **Novel Procedures for Graph Edge-colouring**, após terem inquirido o aluno e realizado a avaliação do trabalho, são de parecer pela sua APROVAÇÃO no rito de defesa.
A outorga do título de doutor está sujeita à homologação pelo colegiado, ao atendimento de todas as indicações e correções solicitadas pela banca e ao pleno atendimento das demandas regimentais do Programa de Pós-Graduação.

Curitiba, 05 de Dezembro de 2018.

RENATO JOSÉ DA SILVA CARMO
Presidente da Banca Examinadora

ANDRÉ LUIZ PIRES GUEDES
Avaliador Interno (UFPR)

RAPHAEL MACHADO
Avaliador Externo (UFRJ)

CELINA MIRAGLIA HERRERA DE FIGUEIREDO
Avaliador Externo (UFRJ)

MURILO VICENTE GONÇALVES DA SILVA
Avaliador Interno (UFPR)

MINISTÉRIO DA EDUCAÇÃO
UNIVERSIDADE FEDERAL DO PARANÁ
Setor de Ciências Exatas
Coordenação de Pós-Graduação em Informática

To Giullia and Arthur,
the colours of my life
in an uncoloured world

Acknowledgements

I would like to sincerely and most heartily express my gratitude:

- to my beloved wife, Giullia, and my dear son, Arthur, for standing by my side along this journey, and for all their sacrifice;
- to my family, specially my parents, my mother-in-law, and my brother and sister, for all the support provided;
- to my advisor, Dr Renato Carmo, and my co-advisor, Dr André Luiz P. Guedes, for the time and the effort which they have dedicated to my doctorate, for all that I have learned with them, and also for the great friendship;
- to Dr Sheila M. Almeida, Alesom Zorzi, and João Pedro W. Bernardi, for the research partnership, which led to some of the achieved results, and also for the valuable suggestions given;
- to the invited professors in the examination board, Dr Celina M. H. Figueiredo, Dr Raphael S. Machado, and Dr Murilo V. G. Silva, for the careful reading and the valuable suggestions;
- to the Federal University of Fronteira Sul (UFFS), for granting me the full-paid leave of absence (grant no 23205.001243/2016-30);
- to the professors and students at UFFS, for all the support, specially to the temporary professors who replaced me and assumed the teaching assignments;
- to the Federal University of Paraná (UFPR), specially its Graduate Programme in Informatics, for the acceptance of my doctoral proposal, for the provided courses, and also for the partial funding;
- to the National Council for Scientific and Technological Development (CNPq) and to the Alberto Luiz Coimbra Institute for Graduate Studies and Research in Engineering of Federal University of Rio de Janeiro (COPPE/UFRJ), for (through CNPq grant no 428941/2016-8 and PESC/CAPES grant no 0588/2018, respectively) providing me financial support at some of the conferences attended;
- to the Institute of Mathematics and Statics of University of São Paulo (IME/USP), for funding my attendance at São Paulo School of Advanced Science on Algorithms, Combinatorics, and Optimisation, in 2016 (through FAPESP grant no 2015/19068-0);
- to my colleagues from the Algorithms Research Group (ARG) of UFPR and to all my close friends, for the support and the great time which we had together;
- to Him, from Who, through Who, and for Who are all things.

Colour is my day-long obsession, joy, and torment.

— Claude Monet

Abstract

The chromatic index of a graph G is the minimum number of colours needed to colour the edges of G in a manner that no two adjacent edges receive the same colour. By the celebrated Vizing's Theorem, the chromatic index of any simple graph G is either its maximum degree Δ or it is $\Delta + 1$, in which case G is said to be *Class 1* or *Class 2*, respectively. Computing an optimal edge-colouring of a graph or simply determining its chromatic index are important $\mathcal{NP}$-hard problems which appear in noteworthy applications, like sensor networks, optical networks, production control, and games. In this work we present novel polynomial-time procedures for optimally edge-colouring graphs belonging to some large sets of graphs. For example, let $\mathscr{X}$ be the class of the graphs whose majors (vertices of degree Δ) have local degree sum at most $\Delta^2 - \Delta$ (by 'local degree sum' of a vertex x we mean the sum of the degrees of the neighbours of x). We show that almost every graph is in $\mathscr{X}$ and, by extending the recolouring procedure used by Vizing's in the proof for his theorem, we show that every graph in $\mathscr{X}$ is *Class 1*. We further achieve results in other graph classes, such as join graphs, circular-arc graphs, and complementary prisms. For instance, we show that a complementary prism can be *Class 2* only if it is a regular graph distinct from the K_2. Concerning join graphs, we show that if G_1 and G_2 are disjoint graphs such that $|V(G_1)| \leqslant |V(G_2)|$ and $\Delta(G_1) \geqslant \Delta(G_2)$, and if the majors of G_1 induce an acyclic graph, then the join graph $G_1 * G_2$ is *Class 1*. Besides these results on edge-colouring, we present partial results on total colouring join graphs, cobipartite graphs, and circular-arc graphs, as well as a discussion on a recolouring procedure for total colouring.

Keywords: Colouring of graphs and hypergraphs (MSC 05C15). Graph algorithms (MSC 05C85). Graph theory in relation to Computer Science (MSC 68R10). Vertex degrees (MSC 05C07). Graph operations (MSC 05C76)..

Resumo

O índice cromático de um grafo G é o menor número de cores necessário para colorir as arestas de G de modo que não haja duas arestas adjacentes recebendo a mesma cor. Pelo célebre Teorema de Vizing, o índice cromático de qualquer grafo simples G ou é seu grau máximo Δ, ou é $\Delta + 1$, em cujo caso G é dito *Classe 1* ou *Classe 2*, respectivamente. Computar uma coloração de arestas ótima de um grafo ou simplesmente determinar seu índice cromático são problemas $\mathcal{NP}$-difíceis importantes que aparecem em aplicações notáveis, como redes de sensores, redes ópticas, controle de produção, e jogos. Neste trabalho, nós apresentamos novos procedimentos de tempo polinomial para colorir otimamente as arestas de grafos pertences a alguns conjuntos grandes. Por exemplo, seja $\mathscr{X}$ a classe dos grafos cujos maiorais (vértices de grau Δ) possuem soma local de graus no máximo $\Delta^2 - \Delta$ (entendemos por 'soma local de graus' de um vértice x a soma dos graus dos vizinhos de x). Nós mostramos que quase todo grafo está em $\mathscr{X}$ e, estendendo o procedimento de recoloração que Vizing usou na prova para seu teorema, mostramos que todo grafo em $\mathscr{X}$ é *Classe 1*. Nós também conseguimos resultados em outras classes de grafos, como os grafos-junção, os grafos arco-circulares, e os prismas complementares. Como um exemplo, nós mostramos que um prisma complementar só pode ser *Classe 2* se for um grafo regular distinto do K_2. No que diz respeito aos grafos-junção, nós mostramos que se G_1 e G_2 são grafos disjuntos tais que $|V(G_1)| \leqslant |V(G_2)|$ e $\Delta(G_1) \geqslant \Delta(G_2)$, e se os maiorais de G_1 induzem um grafo acíclico, então o grafo-junção $G_1 * G_2$ é *Classe 1*. Além desses resultados em coloração de arestas, apresentamos resultados parciais em coloração total de grafos-junção, de grafos arco-circulares, e de grafos cobipartidos, bem como discutimos um procedimento de recoloração para coloração total.

Palavras-chave: Coloração de grafos e hipergrafos (MSC 05C15). Algoritmos de grafos (MSC 05C85). Teoria dos grafos em relação à Ciência da Computação (MSC 68R10). Graus de vértices (MSC 05C07). Operações de grafos (MSC 05C76)..

List of Figures

List of Symbols

$\alpha(G)$	independent set number of the graph G (see p. 35)
B_G	the complete bipartite graph defined by $B_G := G - (E(G_1) \cup E(G_2))$, being G the join graph $G_1 * G_2$ (see p. 69)
B_G	the (not necessarily complete) bipartite graph defined by $B_G := G[\partial_G(V_1)] = G[\partial_G(V_2)]$, being G a cobipartite graph and V_1 and V_2 two disjoint cliques which partition $V(G)$ (see p. 91)
$\chi(G)$	chromatic number of the graph G (see p. 30)
$\chi'(G)$	chromatic index of the graph G (see p. 12)
$\chi''(G)$	total chromatic number of the graph G (see p. 89)
$\mathrm{ch}(G)$	vertex-choosability of the graph G (see p. 90)
$\mathrm{ch}'(G)$	edge-choosability of the graph G (see p. 90)
C_n	the n-length cycle (see p. 15)
$\mathrm{co}\mathcal{NP}$	class of the complements of the decision problems in $\mathcal{NP}$ (see p. 38)
$\Delta(G)$	maximum degree of the graph G (see p. 15)
$\delta(G)$	minimum degree of the graph G (see p. 15)
$d(G)$	the degree of every vertex in the regular graph G (see p. 15)
$d_G(u)$	degree of the vertex u in the graph G (see p. 15)
$\partial_G(u)$	set of edges incident to the vertex u in the graph G (see p. 14)
$\partial_G(X)$	cut defined by the set $X \subseteq V(G)$ in the graph G (see p. 15)
$E(G)$	set of edges of the graph G (see p. 12)
$\overline{G}$	complement of the graph G (see p. 14)
$G_1 * G_2$	the join of the graph G_1 with the graph G_2 (see p. 25)
GG	the prism graph obtained from G (see p. 26)
$G\overline{G}$	the complementary prism graph obtained from G (see p. 26)
G_M	the graph defined by $G_M := (G_1 \cup G_2) + M$ for the maximal matching M on B_G, being G the join graph $G_1 * G_2$ (see p. 69)
$\mathscr{G}(n,p)$	the random graph model on n vertices such that the probability that there is an edge between a pair of distinct vertices is p (see p. 25)
$G_\varphi[\alpha,\beta]$	the graph on the same vertex set as G whose edges are the edges of G coloured α or β by the edge-colouring φ (see p. 18)
$G[X]$	subgraph of G induced by X (see p. 16)
K_n	the complete graph on n vertices (see p. 14)
K_{n_1,n_2}	complete bipartite graph with n_1 and n_2 vertices in each part (see p. 16)
$\Lambda[G]$	core of the graph G (see p. 16)
$\boldsymbol{\Lambda}[G]$	hard core of the graph G (see p. 60)
$\mathbb{\Lambda}[G]$	soft core of the graph G (see p. 60)
$\cancel{\Lambda}[G]$	semi-core of the graph G (see p. 34)

$\mathbf{A}[G]$	hard semi-core of the graph G (see p. 61)
$\mathrm{L}(G)$	line graph of the graph G (see p. 31)
$\mu(\mathcal{G})$	multiplicity of the multigraph $\mathcal{G}$ (see p. 22)
$\nu(G)$	matching number of the graph G (see p. 35)
$N_G(u)$	set of neighbours of the vertex u in the graph G (see p. 14)
$\mathcal{NP}$	class of the decision problem whose positive instances have polynomial-length certificates which can be verified in polynomial time (see p. 37)
$\omega(G)$	clique number of the graph G (see p. 35)
$\mathcal{P}$	class of the decision problems decidable in polynomial time (see p. 37)
P^*	the Petersen graph minus any vertex (see p. 44)
P_n	the $(n-1)$-length path (see p. 15)
ψ_1	canonical edge-colouring of a complete graph of odd order (see p. 17)
ψ_2	canonical edge-colouring of a complete graph of even order (see p. 17)
$\mathrm{T}(G)$	total graph of the graph G (see p. 98)
$V(G)$	set of vertices of the graph G (see p. 12)
$\mathscr{X}$	the class of the graphs with maximum degree Δ whose majors have local degree sum at most $\Delta^2 - \Delta$ (see p. 24)

Contents

1 Introduction

The Wonderland of Graph Theory is full of wonders to be found and explored. In order to map these wonders and catch them all, our fictional character Alice has spread several sensors all over the land. A pair of sensors communicate by radio, sending and receiving messages to and from each other, as long as the distance between them is at most d (see Figure 1.1). A sensor cannot communicate with two other sensors

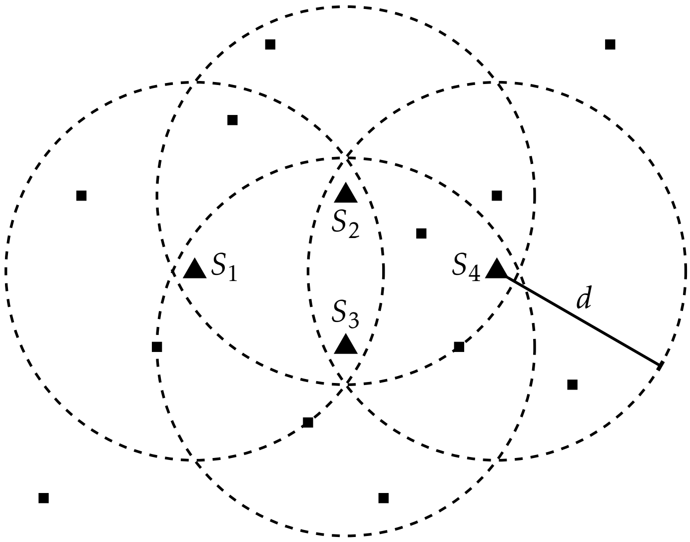


Figure 1.1: A sample of the sensors (represented by filled triangles) which Alice has spread over the Wonderland of Graph Theory in order to map the wonders on the land (represented by small filled squares). The transmission range of a sensor S is depicted by a dashed circle of radius d centred at S. Observe that (S_1, S_4) is the only pair of non-adjacent sensors in this sample.

at the same time, otherwise the overlapping messages may become indistinguishable. Therefore, Alice wants to program the sensors to change frequency tuning at every period of time t, in a manner that each sensor communicates with its adjacent sensors one at a time and, after a certain amount k of these periods, every pair of adjacent sensors have communicated. Of course, Alice wants k to be as small as possible.

Alice's problem of minimising k is used in this chapter to introduce the subject of our study in this work.

1.1 Graphs and edge-colouring

We can model Alice's problem presented above using an abstract structure known as *graph*. A *graph* is a mathematical object consisting of *vertices* and connections between them, called *edges*[1]. There are several types of graphs, but in this work we concern ourselves mainly with *simple graphs*, that is, graphs wherein the edges are undirected connections between two distinct vertices. In the example of Alice's problem,

[1] For more information on graph-theoretical concepts, results, and applications, we refer the reader to Bondy and Murty (2008) and Diestel (2010).

we represent each sensor by a vertex and we create an edge between two sensors if and only if they can communicate (see Figure 1.2).

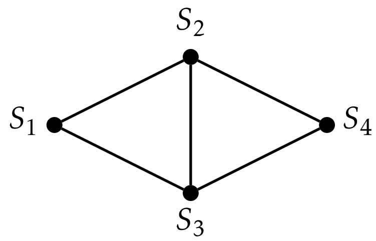


Figure 1.2: A graph representing the adjacencies between the sensors depicted in Figure 1.1. Particularly, this graph is known as the *diamond graph* .

Formally, we can define a graph as a pair $G = (V, E)$, wherein V is the finite set of vertices of G and each element e of E is a set $\{u, v\}$, often simply denoted uv, for two distinct u and v in V. Whenever the *set of vertices* and the *set of edges* of G are not specified, they are simply referred to as $V(G)$ and $E(G)$, respectively. The number of vertices of a graph is referred to as its *order*.

Considering a graph such as the one shown in Figure 1.2, let us assign to each edge uv of the graph a number i, indicating that the sensors represented by the vertices u and v communicate during the i^{th} period of time t (see Figure 1.3). Observe that,

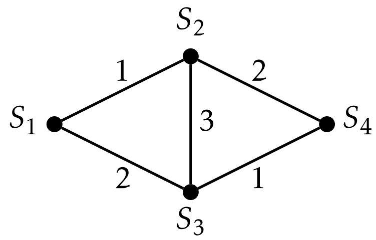


Figure 1.3: An edge-colouring of the graph of Figure 1.2. Observe that the number of colours used, 3, is minimal, since there are 3 edges incident to S_2 and each one needs to receive a distinct colour.

for this assignment, only the structure of the graph itself is relevant, regardless of *what* is being modelled by the vertices and the edges. In fact, many other application problems[2], from very different contexts, can be viewed as the problem of assigning numbers (hereinafter referred to as *colours*) to the edges of a graph in a manner that no two adjacent edges are assigned the same colour and the least amount of different colours is used. This is the problem of computing an optimal *edge-colouring* of a graph.

A *k-edge-colouring* of a graph G is a function $\varphi\colon E(G) \to \mathscr{C}$ such that $\mathscr{C}$ is a set with k colours and $\varphi(e) \neq \varphi(f)$ whenever e and f are distinct adjacent edges. The least k for which G is k-edge-colourable is the *chromatic index* of G, denoted $\chi'(G)$. For instance, the graph of Figure 1.3 has chromatic index 3, and the graph of Figure 1.4 on the next page has chromatic index 4.

Computing an optimal edge-colouring of a graph or simply computing its chromatic index are very important graph-theoretical problems, leading to open questions which have been studied through the last 50 years. As discussed in Chapter 2, some of these questions are closely related to central problems in Computer Science about the relations between the main computational complexity classes. The present work aims to contribute in gathering more knowledge about these open questions.

[2]See Section 1.2 for some of these problems.

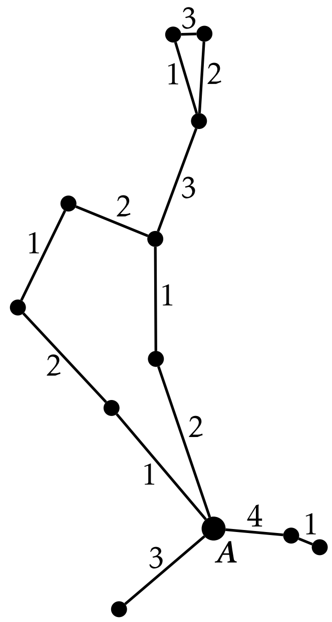


Figure 1.4: A 4-edge-colouring of the graph which models the constellation of Boötes. Since there are 4 edges incident to the vertex $\boldsymbol{A}$, which represents the giant star Arcturus, this edge-colouring is optimal.

1.2 Edge-colouring in real-world applications

In addition to the theoretical motivation behind edge-colouring problems, their importance and relevance are also justified by some real-world applications. Alice's problem described in the beginning of this chapter is actually a simplification of the problem of link scheduling by communication protocols in sensor networks (Gandham et al., 2005). A pair of Alice's sensors can send and receive messages to and from each other at the same time, something that usually cannot happen with real-world sensors. Even so, computing an edge-colouring of the corresponding graph may be an important step to compute a feasible link schedule for a sensor network. For instance, Gandham et al. (2005) present a distributed algorithm which first computes a near-optimal edge-colouring (note the importance of distributedness for the context of sensor networks). Then, once the edge-colouring has been computed, they show how to obtain a "TDMA MAC schedule which enables two-way communication between every pair of neighbours" (Ibid.) using *at most* twice as many time slots as the number of colours used if the graph is acyclic, or "*approximately* twice" if the graph is sparse.

Still in the context of communication protocols, edge-colouring appears interestingly in some challenging problems concerning modern high-performance networks. Consider, for example, the problem of assigning wavelengths to connection requests in optical networks with wavelength-division multiplexing, minimising the total amount of wavelengths used. In this problem, each connection request must be assigned a path between the requesting pair of nodes and a wavelength for the transmission, in a manner that no two connection requests whose paths intersect are assigned the same wavelength. This can be modelled as the graph-theoretical problem of computing an optimal *path-colouring* of the corresponding graph G (Erlebach and Jansen, 2001), which is equivalent to the problem of computing an optimal edge-colouring of an associated multigraph when G is an undirected tree (Ibid.).

Other applications on scheduling have been explored, like process scheduling in production control (Williamson et al., 1997) and match scheduling in games (Burke et al., 2004, Sect. 5.6; Skiena, 2008, Sect. 16.8; Januario et al., 2016). As a particular case of the latter, consider the problem of organising a tournament with n teams which are supposed to play against each other pairwise, in a manner that no team can play in more than one match per day and the tournament lasts the least as possible. This problem can be modelled as the problem of edge-colouring complete graphs. The *complete graph* on n vertices, denoted K_n, is the n-vertex graph wherein all vertices are pairwise adjacent (see Figure 1.5). As it shall be demonstrated in Section 1.4, one of the earliest results on edge-colouring is that $\chi'(K_n) = n - 1$ if n is even, or $\chi'(K_n) = n$ if n is odd.

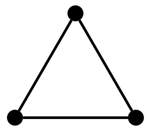


Figure 1.5: The K_3, also known as *triangle*

1.3 Some basic graph-theoretical concepts

We have already defined in Section 1.2 terms such as *graph*, *vertex*, *edge*, and *edge-colouring*. Before we can continue our discussion, we briefly introduce some other necessary basic graph-theoretical concepts in this section, which the reader may choose to skip. Other definitions appear in other parts of this document, preferably presented through the text as soon as they are required. It is important to remark that exhaustively covering all classical graph-theoretical concepts is not the purpose of this work. Therefore, terms and notation usage which are not explicitly defined anywhere should always be assumed to follow their meanings in the textbooks of Bondy and Murty (2008) and Diestel (2010).

Although this work deals mainly with simple graphs, in some results we also use multigraphs. In a *multigraph*, there may be multiple — or *parallel* — edges between a pair of vertices (see Figure 1.6). The multigraphs considered in this work are always

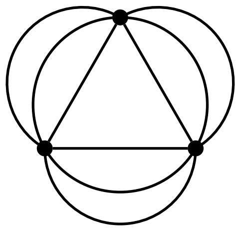


Figure 1.6: The Shannon multigraph

loopless and undirected, which means that we can define formally a multigraph as a pair $\mathcal{G} = (V, E)$ wherein V is a finite set and E is a finite multiset, being each element of E an edge uv with $u, v \in V$. Throughout this text, we shall use the term *graph* to refer always to a simple graph, in contrast to the term *multigraph*. Other graph-theoretical concepts are defined for multigraphs analogously to their corresponding concepts for graphs and follow the same notation.

Let G be a graph. The *complement* of G, denoted $\overline{G}$, is the graph defined by $V(\overline{G}) := V(G)$ and $E(\overline{G}) := \{uv : u, v \in V(G), u \neq v, \text{ and } uv \notin E(G)\}$. Now, let $u \in V(G)$. The *set of neighbours* of u in G (i.e. the set of vertices adjacent to u in G) and the *set* *of edges* incident to u in G are denoted $N_G(u)$ and $\partial_G(u)$, respectively. The number of elements in $\partial_G(u)$ is the *degree* of u in G and is denoted $d_G(u)$. Also, for any $\emptyset \neq X \subsetneq V(G)$, the set $\partial_G(X) := \{uv \in V(G) : u \in X \text{ and } v \notin X\}$ is the *cut* induced by X in G. If $X = V(G)$, we define $\partial_G(X) := \emptyset$, although we do not call this set a *cut*. The vertex u is said to be *universal* in G if $N_G(u) \cup \{u\} = V(G)$.

The *maximum degree* of any vertex of a graph G is denoted $\Delta(G)$, or simply Δ when free of ambiguity. The *minimum degree* of G is denoted $\delta(G)$, or simply δ. The vertices of degree Δ in G are the *majors* of G. For example, in the graph of Figure 1.4 (p. 13), the only major is the vertex marked $\boldsymbol{A}$. If all vertices of a graph G have the same degree d, then G is said to be *d-regular*, or simply regular, being $d(G) := d$. A graph is said to be *cubic* if it is 3-regular, like the Petersen graph (Figure 1.7).

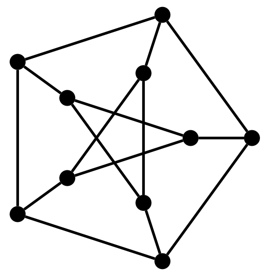


Figure 1.7: The Petersen graph

Let n be a positive integer. The *path* with *n vertices*, denoted P_n, is the graph defined by $V(P_n) := \{u_0, \dots, u_{n-1}\}$ and $E(P_n) := \{u_i u_{i+1} : 0 \leqslant i < n-1\}$, in which case we refer to u_0 and u_{n-1} as the *outer vertices* of the path, as well as to $u_1, \dots, u_{n-2}$ as the *inner vertices*. The *length* of the path P_n is its number of edges, that is, $n-1$. The *cycle* of *length n*, denoted C_n and defined only if $n \geqslant 3$, is the graph defined by $V(C_n) := \{u_0, \dots, u_{n-1}\}$ and $E(C_n) = \{u_i u_{(i+1) \bmod n} : 0 \leqslant i \leqslant n-1\}$. A cycle is said to be *even* (resp. *odd*) if, of course, its length is *even* (resp. *odd*).

A *homomorphism* from a graph G_1 to a graph G_2 is a function $\lambda\colon V(G_1) \to V(G_2)$ such that $\lambda(u)\lambda(v) \in E(G_2)$ for all $uv \in E(G_1)$. If λ is also bijective and $\lambda(u)\lambda(v) \notin E(G_2)$ for all $uv \notin E(G_1)$, then λ is said to be an *isomorphism* between G_1 and G_2, and the graphs G_1 and G_2 are said to be *isomorphic* (see Figure 1.8). If λ is an isomorphism

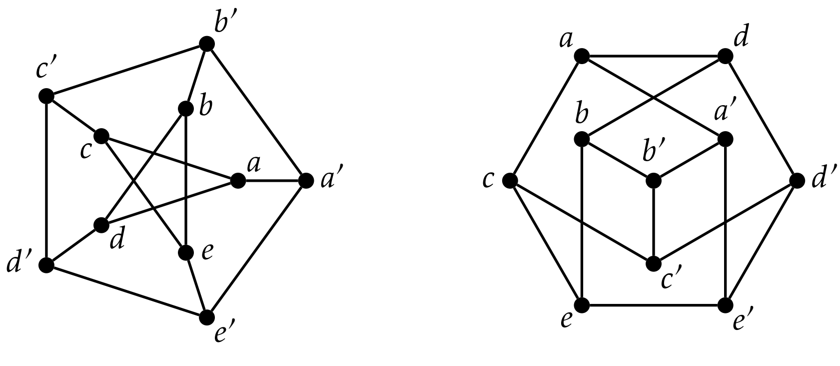


Figure 1.8: A labelling of the vertices of two graphs which identifies their isomorphism

and $G_1 = G_2 =: G$, then λ is said to be an *automorphism* on G. A graph G is said to be *symmetric* if, for all uv and all xy in $E(G)$, there is an *automorphism* λ on G such that $\lambda(u) = x$ and $\lambda(v) = y$. The Petersen graph is an example of a symmetric graph.

A graph H is said to be a *subgraph* of a graph G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. A *spanning* subgraph of G is a subgraph H of G such that $V(H) = V(G)$. A Δ-*subgraph* of G is a subgraph of G with the same maximum degree as G. The subgraph of G *induced* by some $U \subseteq V(G)$, denoted $G[U]$, is the graph defined by $V(G[U]) := U$ and $E(G[U]) := \{uv \in E(G) : u, v \in U\}$. The *core* of G, denoted $\Lambda[G]$, is the subgraph of G induced by its majors. The subgraph of G induced by some $F \subseteq E(G)$, denoted $G[F]$, is defined by $E(G[F]) := F$ and $V(G[F]) := \{u \in V(G) : uv \in F$ for some $v \in V(G)\}$. An *induced* H in G is a subgraph of G induced by some $U \subseteq V(G)$ and isomorphic to H. The graph G is said to be H-free if it has no induced H. For example, the Petersen graph (Figure 1.7) is P_k-free for all $k \geqslant 8$, and all its induced cycles have length 5 or 6, although it contains non-induced cycles of greater length. When we say that $v_1 \cdots v_k$ is a cycle (path) in G, it does not necessarily mean that this cycle (path) is induced.

A *clique* on a graph G is a set $K \subseteq V(G)$ which induces a complete graph. An *independent set* on G is a set $I \subseteq V(G)$ which induces an edgeless graph. A graph G is said to be *bipartite* if $V(G)$ can be partitioned in two non-empty independent sets A and B, called the *parts* of the bipartition. A standard result is that a graph is bipartite if and only if it has no odd cycle (Bondy and Murty, 2008; Diestel, 2010). Given two positive integers r and s, the *complete bipartite graph* $K_{r,s}$ is the bipartite graph with r vertices in A and s in B such that $ab \in E(K_{r,s})$ for all $a \in A$ and all $b \in B$. A *star* is a $K_{1,n}$ for some positive integer n.

A graph G with $V(G) \neq \emptyset$ is said to be *connected* if there is a path between any pair of vertices, and *biconnected* if $|V(G)| > 2$ and if G and $G - u$ are connected for all $u \in V(G)$. The *connected components* (or simply *components*) of G are its maximal connected subgraphs. The *biconnected components* of G are its maximal biconnected subgraphs. If $x \in V(G) \cup E(G)$ and $G - x$ has more components than G, then x is said to be: an *articulation point* (also *cut vertex*) if $x \in V(G)$; a *bridge* (also *cut edge*) if $x \in E(G)$. A *tree* is a connected graph with no cycles, hence all trees are bipartite. A *forest* is a disjoint union of trees.

1.4 Edge-colouring complete graphs and bipartite graphs

As we further discuss in Chapter 2, computing an optimal edge-colouring of a graph or simply its chromatic index are $\mathcal{NP}$*-hard* problems in general (Holyer, 1981). However, when dealing with graphs which satisfy certain properties, the problems may become *easy* (that is, *polynomial*). For example, the chromatic index of a complete graph can be determined only by the parity of its order, as Theorem 1.2 establishes. Moreover, an optimal edge-colouring of a complete graph can also be computed *efficiently*, since the proof of Theorem 1.2 on the next page is constructive and yields a polynomial-time optimal edge-colouring algorithm.

Before we proceed to Theorem 1.2, we make an observation which shall be used in this proof and in other arguments in this text.

Observation 1.1. *Any graph G on n vertices has at most $\chi'(G)\lfloor n/2 \rfloor$ edges.*

Proof. For any colour α in a k-edge-colouring, if the number of edges coloured α is t, then the number of vertices which are ends of an α-coloured edge is exactly $2t$. Therefore, $2t \leqslant n$, which implies $t \leqslant \lfloor n/2 \rfloor$ (because t is an integer). Summing this inequality over all the k colours, we get $|E(G)| \leqslant k\lfloor n/2 \rfloor$. □

The following is a standard result on edge-colouring shown, for example, by Behzad et al. (1967), but known much earlier.

Theorem 1.2. *The chromatic index of the graph K_n is $n-1$, if n is even, or n, if n is odd.*

Proof. Theorem 1.2 Considering the set of vertices of the K_n as the set of labels $V(K_n) = \{0, 1, \ldots, n-1\}$, we first show that the function $\psi_1 \colon E(K_n) \to \{0, \ldots, n-1\}$ given by

$$\psi_1(uv) := (u+v) \bmod n \tag{1.1}$$

is an n-edge-colouring of the K_n, regardless of the parity of n (see Figure 1.9). In order to demonstrate this, it suffices to prove, for any uv and any uw in $E(K_n)$ such that $\psi_1(uv) = \psi_1(uw)$, that $v = w$. But this follows immediately from the fact that $u + v \equiv u + w \pmod{n}$ implies $v = w$ by elementary modular arithmetic.

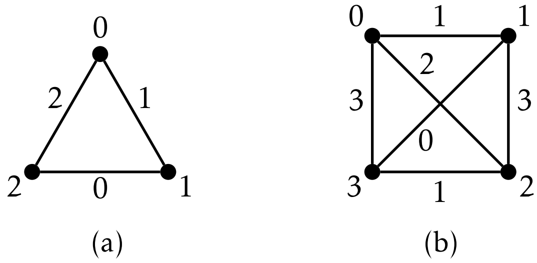


Figure 1.9: The n-edge-colouring of the K_n given by the function ψ_1 in the cases wherein n is (a) 3 and (b) 4. In the former case, the edge-colouring shown is optimal. In the latter, it is not.

We have proved that $\chi'(K_n) \leqslant n$. Now we shall prove that $\chi'(K_n) \geqslant n$ in the case wherein n is odd. We know by Observation 1.1 that $|E(K_n)| \leqslant \chi'(K_n)(n-1)/2$ if n is odd. Since $|E(K_n)| = n(n-1)/2$, this implies, as we wanted,

$$\chi'(K_n) \geqslant \frac{2n(n-1)}{2(n-1)} = n \qquad \text{for odd } n.$$

It remains to show that $\chi'(K_n) = n-1$ when n is even. Since exactly $n-1$ edges are incident to each vertex of the K_n, we know that $\chi'(K_n) \geqslant n-1$. Hence, considering that n is even, it suffices to exhibit an $(n-1)$-edge-colouring of the K_n. We claim that such an edge-colouring is the function $\psi_2 \colon E(K_n) \to \{0, \ldots, n-2\}$ defined by

$$\psi_2(uv) := \begin{cases} (u+v) \bmod (n-1), & \text{if } u, v \in \{0, \ldots, n-2\}, \\ (2u) \bmod (n-1), & \text{if } v = n-1, \\ (2v) \bmod (n-1), & \text{if } u = n-1, \end{cases} \tag{1.2}$$

In order to prove the claim, we take two edges uv and uw of the K_n and we assume $\psi_2(uv) = \psi_2(uw)$. We shall prove that $v = w$. We have the following cases:

1. if $u, v, w \in \{0, \ldots, n-2\}$, then $u + v \equiv u + w \pmod{(n-1)}$, which implies $v = w$;

2. if $u = n-1$, then $v, w \in \{0, \ldots, n-2\}$ and $2v \equiv 2w \pmod{(n-1)}$, which also implies $v = w$ since $n-1$ is odd;

3. if v or w, say v, is $n-1$, and if we assume $v \neq w$, then $u, w \in \{0, \ldots, n-2\}$ and $2u \equiv u + w \pmod{(n-1)}$, which implies $w = u$, a contradiction. □

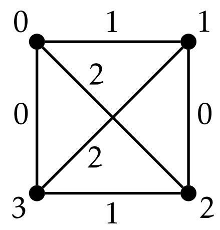


Figure 1.10: The canonical edge-colouring of the K_4

The function ψ_1 (resp. ψ_2) defined in (1.1) (resp. (1.2)) is hereinafter referred to as the *canonical* edge-colouring of the K_n with n odd (resp. even). Figures 1.9(a) and 1.10 show the canonical edge-colourings of the K_3 and the K_4, respectively.

It is clear that $\chi'(G) \geqslant \Delta(G)$ for any graph G, since each edge incident to a vertex must receive a distinct colour (recall Figure 1.3, p. 12). Moreover, this bound is tight, since many (but not all) graphs satisfy $\chi'(G) = \Delta(G)$. The classical result transcribed in Theorem 1.3 brings that this equality holds for all bipartite graphs.

Theorem 1.3 (Kőnig's Theorem (Kőnig, 1916 *apud* Stiebitz et al., 2012)). *The chromatic index of any bipartite graph is equal to its maximum degree.*

In the proof of Theorem 1.3, and in many other proofs in this document, being $\varphi\colon E(G) \to \mathscr{C}$ an edge-colouring of any (not necessarily bipartite) graph G, we say that some colour $\alpha \in \mathscr{C}$ is *missing at* some $u \in V$, and that u *misses* α, if no edge incident to u is coloured α. Also, being $\beta \in \mathscr{C} \setminus \{\alpha\}$, we use $G_\varphi[\alpha,\beta]$ (or simply $G[\alpha,\beta]$ when free of ambiguity) to denote the subgraph of G induced only by the edges which have been coloured α or β by φ, including as isolated vertices the vertices which miss both α and β. Each component of $G_\varphi[\alpha,\beta]$ is often referred to as an α/β-component. It is straightforward to verify that an α/β-component is always a path or an even cycle (see Figure 1.11), since paths and cycles are the connected graphs with $\Delta \leqslant 2$ and, by Observation 1.1, odd cycles are not 2-edge-colourable. If an α/β-component is a path (resp. an even cycle), it may be referred to as an α/β-path (resp. an α/β-cycle).

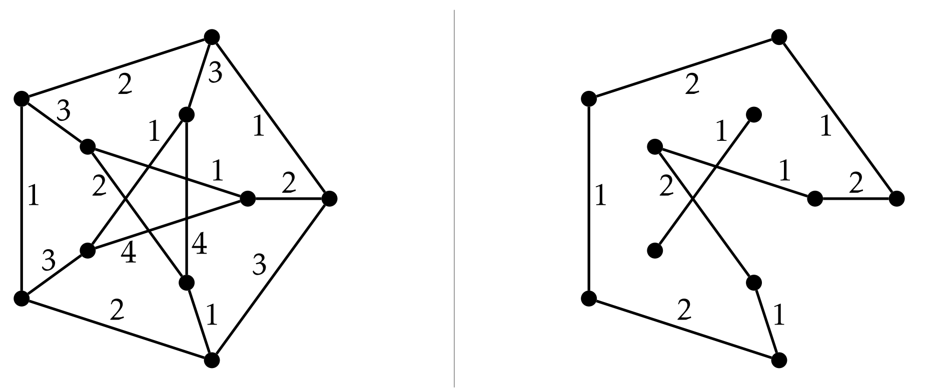


Figure 1.11: A 4-edge-colouring of the Petersen graph using the set of colours $\{1,2,3,4\}$ and the 1/2-components (one path and one cycle) detached. An argument for why the Petersen graph is not 3-edge-colourable shall be discussed in Chapter 2 (Observation 2.2, p. 31).

Proof of Kőnig's Theorem (Theorem 1.3). Let G be a bipartite graph and let $\mathscr{C}$ be a set with Δ colours. We shall construct an edge-colouring $\varphi\colon E(G) \to \mathscr{C}$ edge by edge.

For each edge uv considered, if there is some $\alpha \in \mathscr{C}$ missing at both u and v, then uv can be coloured α and we proceed to the next edge. However, if no colour of $\mathscr{C}$ is missing at both u and v, let α be a colour missing at u and β be a colour missing at v. Since u misses α, the α/β-component to which u belongs is a path P, not a cycle, and u is an outer vertex of this path. Furthermore, v cannot be in P, otherwise P would be an even-order (odd-length) path with u and v as its outer vertices, and $P + uv$ would thus be an odd cycle in G. Ergo, by exchanging the colours along P, we obtain β as a colour missing at both u and v, then the edge uv can be coloured β. □

1.5 Vizing's recolouring procedure

We have shown in Section 1.4 that an optimal edge-colouring of a complete or bipartite graph can be computed efficiently. However, back to Alice's problem described in the beginning of this chapter, it may be the case that her graph G, whose edges she wants to colour, is neither complete nor bipartite. In fact, Alice may have no information whatsoever about the structure of G. She knows only that $\chi'(G) \geqslant \Delta(G)$, since each edge incident to a vertex must receive a distinct colour (recall Figure 1.3, p. 12). Surprisingly, *Vizing's Theorem*, for which we shall transcribe a proof in the sequel, establishes a breakthrough upper bound for $\chi'(G)$ for every (simple) graph G.

Theorem 1.4 (Vizing's Theorem (Vizing, 1964 *apud* Stiebitz et al., 2012)). *The chromatic index of a graph G is not greater than* $\Delta(G) + 1$.

Graphs which have chromatic index equal to its maximum degree Δ, such as the diamond (Figure 1.2, p. 12), the graph of the constellation of Boötes (Figure 1.4, p. 13), and all bipartite graphs (recall Theorem 1.3, p. 18), are called *Class 1*. On the other hand, graphs which do not admit a Δ-edge-colouring, such as the K_n when n is odd (recall Theorem 1.2, p. 17) and the Petersen graph (Figure 1.7, p. 15), have chromatic index equal to $\Delta + 1$ and are called *Class 2*. In view of Vizing's breakthrough, the problem of deciding the chromatic index of a graph is often referred to as the *Classification Problem*, but we avoid this nomenclature in this work.

Vizing's proof for Theorem 1.4 uses a polynomial-time *recolouring procedure* to construct a $(\Delta + 1)$-edge-colouring of any graph edge by edge. Before discussing Vizing's recolouring procedure in details, we highlight a key aspect about Vizing's proof which substantially distinguishes it from the proofs in Section 1.4.

Recall that the proofs in Section 1.4 also yield polynomial-time algorithms to construct the optimal edge-colourings edge by edge. However, at each edge uv considered to receive a colour, we have enough information about the edge-colouring constructed so far to take advantage of it. In the proof for complete graphs, we know that evaluating a simple modular arithmetic expression will not create colour conflicts. In the proof for bipartite graphs, we know from the non-existence of odd cycles that u and v cannot be in the same α/β-path for any α missing at u and any β missing at v.

Differently from the proofs in Section 1.4, Vizing's proof deals with graphs with no particular structure given, wherein almost nothing is known about the edge-colouring φ constructed so far while colouring an edge uv. This is why a *recolouring* procedure plays a most important role in Vizing's proof: it shows that no matter how unfavourable the unknown edge-colouring φ can be, in every possible case we can *recolour* some edges of the graph in order to make φ favourable. This same principle is applied in the proofs of our own presented in this thesis when dealing with graphs with not so much structure given, for which we extend Vizing's recolouring procedure.

Definition 1.5 and Lemmas 1.6–1.9 together present Vizing's recolouring procedure. In this definition and in these lemmas, $G = (V, E)$ is a graph and $\varphi\colon E \setminus \{uv\} \to \mathscr{C}$ is an edge-colouring of $G - uv$ for some $uv \in E$.

Definition 1.5. A sequence $v_0, \dots, v_k$ of *distinct* neighbours of u in G is a *Vizing's recolouring fan* for uv if $v_0 = v$ and, for all $i \in \{0, \dots, k-1\}$, v_i misses the colour $\alpha_i := \varphi(uv_{i+1})$. The fan is said to be *complete* if v_k misses a colour which is also missing at u.

It is important to notice that, since the vertices $v_0, \dots, v_k$ of a recolouring fan are all distinct, the colours $\alpha_0, \dots, \alpha_{k-1}$ are all distinct colours not missing at u.

Lemma 1.6. *If there is a complete Vizing's recolouring fan for uv, then G admits an edge-colouring using only the colours of $\mathscr{C}$.*

Proof. We perform a procedure for i from k down to 0. At the beginning of each iteration it is invariant that both u and v_i miss α_i. So, we simply assign α_i to uv_i. If $i = 0$, we are done. If $i > 0$, now u misses α_{i-1}, which is still missing at v_{i-1}, so we decrement i and continue. □

The procedure described in the proof of Lemma 1.6 is referred to as the *decay of the colours* of the Vizing's recolouring fan $v_0, \dots, v_k$ (see Figure 1.12),

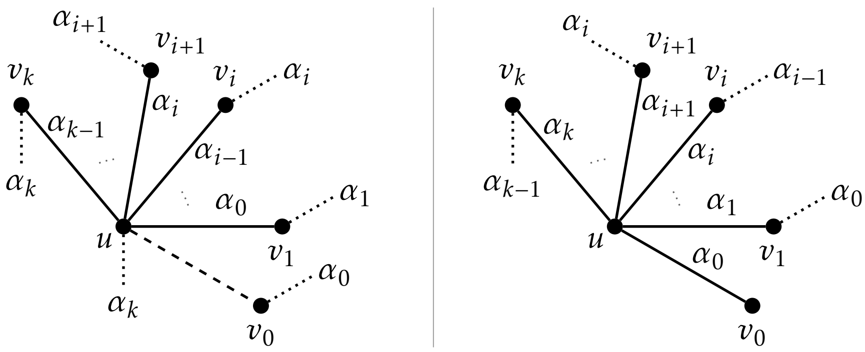


Figure 1.12: A complete Vizing's recolouring fan before and after the decay of the colours. The dotted lines indicate the colours missing at the vertices, and the dashed line the edge to be coloured.

Lemma 1.7. *If there is a Vizing's recolouring fan $v_0, \dots, v_k$ for uv such that, for some $\beta \in \mathscr{C}$ missing at u and some $\alpha_k \in \mathscr{C} \setminus \{\alpha_0, \dots, \alpha_{k-1}, \beta\}$ missing at v_k, the vertices u and v_k are* not *in the same α_k/β-component, then G admits an edge-colouring using only the colours of $\mathscr{C}$.*

Proof. The α_k/β-component to which v_k belongs is a path P which has v_k as one of its outer vertices. Exchanging the colours along P brings β missing at both u and v_k. Since the colours $\alpha_0, \dots, \alpha_k, \beta$ are all distinct, the colour exchanging operation does not compromise the recolouring fan (see Figure 1.13 on the next page). Hence, the sequence $v_0, \dots, v_k$ is now a complete Vizing's recolouring fan for uv, so applying Lemma 1.6 concludes the proof. □

Lemma 1.8. *If there is a Vizing's recolouring fan $v_0, \dots, v_k$ for uv such that v_k misses the colour α_j for some $j < k$, then G admits an edge-colouring using only the colours of $\mathscr{C}$.*

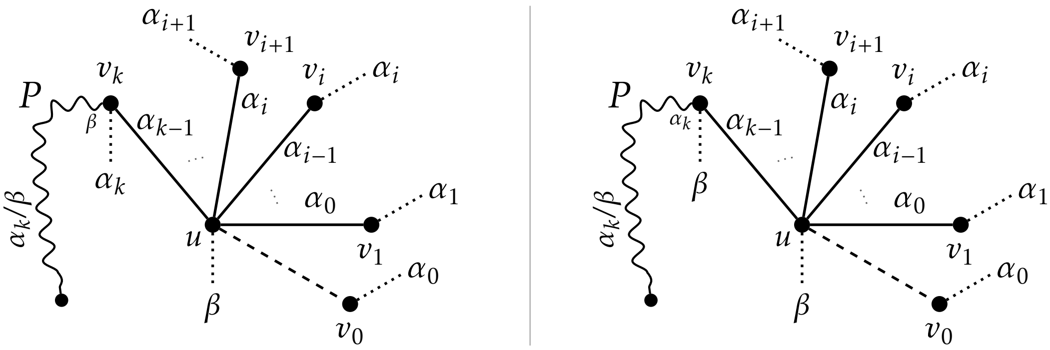
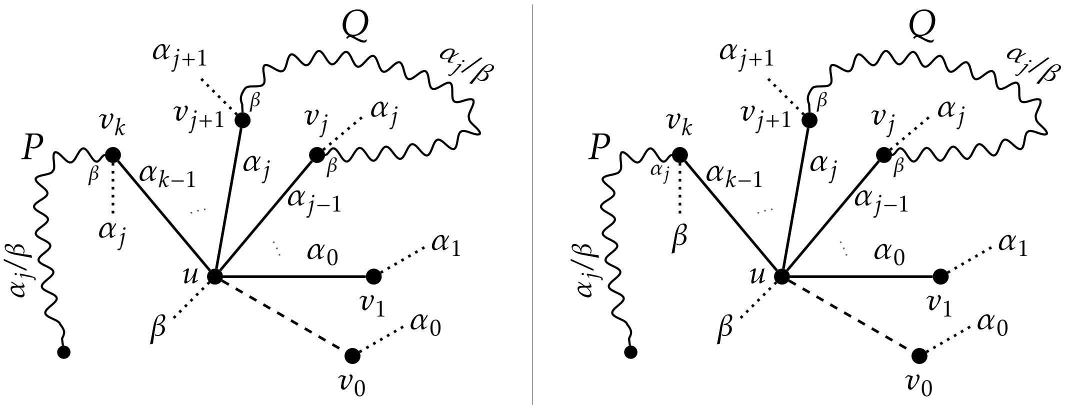


Figure 1.13: A recolouring fan under the conditions of Lemma 1.7 before and after the colour exchanging operation along the α_k/β-path P, which is depicted as a waved curve. The tiny colour name near v_k indicates the colour of the edge of P incident to v_k.

Proof. First remark that $j < k-1$, since $\alpha_{k-1} = \varphi(uv_k)$. If the recolouring fan $v_0,\ldots,v_j$ satisfies Lemma 1.7, we are done. Otherwise, let β be a colour of $\mathscr{C}$ missing at u and let P (clearly a path) be the α_j/β-component to which v_k belongs. We know that u and v_j are in the same α_j/β-component, which is a path Q disjoint from P having u and v_j as outer vertices (see Figure 1.14(a)), otherwise Lemma 1.7 would have been satisfied by $v_0,\ldots,v_j$. Hence, exchanging the colours of the edges along P yields β as a colour



(a) (b)

Figure 1.14: An illustration for the proof of Lemma 1.8

missing at both u and v_k, and α_j as a colour still missing at v_j (see Figure 1.14(b)). Because the colours $\alpha_0,\ldots,\alpha_{k-1},\beta$ are all distinct, now $v_0,\ldots,v_k$ is a complete Vizing's recolouring fan for uv, so the proof is concluded by Lemma 1.6. □

Lemma 1.9. *If all the neighbours of u in G miss at least one colour of $\mathscr{C}$ each, then G admits an edge-colouring using only the colours of $\mathscr{C}$.*

Proof. First observe that the unitary sequence $v(=: v_0)$ is a Vizing's recolouring fan for uv. If this recolouring fan is complete (that is, if there is some colour missing at both u and v), we are done by Lemma 1.6. Otherwise, since u has finitely many neighbours, each of which misses at least one colour of $\mathscr{C}$, we can continue to construct the recolouring fan until we reach a fan $v_0,\ldots,v_k$ such that v_k misses some colour also missing at u, satisfying Lemma 1.6, or v_k misses α_j for some $j < k$, satisfying Lemma 1.8. Then, we apply the corresponding proof in order to colour uv. □

By Lemma 1.9, Vizing's Theorem follows immediately.

Proof of Vizing's Theorem (Theorem 1.4, p. 19). Let $\mathscr{C}$ be a set with $\Delta + 1$ colours. We shall construct an edge-colouring $\varphi\colon E(G) \to \mathscr{C}$ edge by edge. The proof follows by observing that we are using more colours than the degree of any vertex and hence, at each edge uv considered, the edge uv satisfies the condition of Lemma 1.9. □

We remark that our exposition of Vizing's recolouring procedure, which deals only with simple graphs, is slightly simpler than Vizing's original exposition, which also considers multigraphs. In fact, with his procedure Vizing showed that the chromatic index of any multigraph $\mathcal{G}$ is not greater than $\Delta(\mathcal{G}) + \mu(\mathcal{G})$, being $\mu(\mathcal{G})$ the *multiplicity* of $\mathcal{G}$, i.e. the largest number of edges with the same pair of end-vertices, is denoted $\mu(\mathcal{G})$. Observe that this upper bound is tight: an example of a multigraph $\mathcal{G}$ with chromatic index exactly $\Delta(\mathcal{G}) + \mu(\mathcal{G})$ is the Shannon multigraph (Figure 1.6, p. 14).

1.6 Main contribution and structure of this work

We present novel polynomial-time procedures for optimally edge-colouring large sets of graphs. Our main Δ-edge-colourability proofs are based on new recolouring procedures which extend Vizing's recolouring procedure discussed in Section 1.5. We close this introductory chapter by announcing the main results achieved by this work and showing how this document is organised. Along with the main theorems stated in this section, other results can be found in the chapters that follow.

Since Vizing showed that there can be only two possibilities for the chromatic index of any graph, and since Holyer (1981) proved that deciding between these two possibilities is an $\mathcal{NP}$-hard problem for graphs in general, there has been much work aimed at determining the computational complexity of restrictions of the problem. Table 1.1 shows some graph classes in which the problem has been already proved to be polynomial or $\mathcal{NP}$-hard, as well as some graph classes in which the complexity remains open, with only partial results being known.

Table 1.1: Some results from the literature about the computational complexity of the problem of deciding the chromatic index of an n-vertex graph belonging to some restricted graph class. For classes wherein the problem remains open, references are provided as long as they contain further partial, but too technical, results than the results listed for the subclasses, if any. Some results are covered by others, but are displayed nevertheless for historical purposes. Classes which are entirely contained in more than one of the listed classes are displayed only once. For example, indifference graphs appear only below chordal graphs, although they are also a subclass of dually chordal graphs.

Graph class	Complexity	References
bipartite graphs	polynomial	Kőnig (1916 *apud* Stiebitz et al., 2012)
clique graphs	$\mathcal{NP}$-hard	Cai and Ellis (1991)
↳ line graphs of bipartite graphs	$\mathcal{NP}$-hard	Cai and Ellis (1991)
↳ dually chordal graphs	open	
↳ with odd Δ	polynomial	Figueiredo et al. (1999)
cobipartite graphs	open	Machado and Figueiredo (2010)
d-regular graphs	$\mathcal{NP}$-hard	see subclasses
↳ with odd n	polynomial	standard, see Chapter 2
↳ with $d = 3$	$\mathcal{NP}$-hard	Holyer (1981)
↳ which are triangle-free	$\mathcal{NP}$-hard	Cai and Ellis (1991); Koreas (1997)

Table 1.1: Some results from the literature about the computational complexity of the problem of deciding the chromatic index of an n-vertex graph belonging to some restricted graph class (continued)

Graph class	Complexity	References
↳ with any constant $d \geqslant 3$	$\mathcal{NP}$-hard	Leven and Galil (1983)
↳ which are k-partite for any constant $k \geqslant 3$	$\mathcal{NP}$-hard	Machado et al. (2010)
↳ with girth at least k for any constant $k \geqslant 3$	$\mathcal{NP}$-hard	Cai and Ellis (1991)
↳ with $d \geqslant n/2$	open	
↳ with $d \geqslant (6/7)n \geqslant 0.86n$	polynomial	Chetwynd and Hilton (1985)
↳ with $d \geqslant ((\sqrt{7}-1)/2)n \geqslant 0.83n$	polynomial	Chetwynd and Hilton (1989a)
↳ with $d \geqslant (1/2+\varepsilon)n$ for any $\varepsilon > 0$ and sufficiently large n	polynomial	Perkovic and Reed (1997)
graphs with $\Delta > n/3$	open	Chetwynd and Hilton (1986) Hilton and Johnson (1987)
↳ with $\Delta \geqslant n/2$	open	Chetwynd and Hilton (1984a)
↳ with $\Delta \geqslant n-3$	polynomial	
↳ with $\Delta \geqslant n-1$	polynomial	Plantholt (1981)
↳ with $\Delta \geqslant n-2$ and even n	polynomial	Plantholt (1983)
↳ with $\Delta \geqslant n-2$ and odd n	polynomial	Chetwynd and Hilton (1984b)
↳ with $\Delta \geqslant n-3$ and even n	polynomial	Chetwynd and Hilton (1984a)
↳ with $\Delta \geqslant n-3$ and odd n	polynomial	Chetwynd and Hilton (1989b)
partial k-trees (i.e. graphs with treewidth at most k) for any constant $k \geqslant 1$	polynomial	Bodlaender (1990)
↳ k-outerplanar graphs for any constant $k \geqslant 1$	polynomial	Bodlaender (1990)
↳ series-parallel graphs	polynomial	Johnson (1985)
↳ cacti (i.e. chordless graphs)	polynomial	Machado et al. (2013)
join graphs	open	De Simone and Mello (2006) De Simone and Galluccio (2009, 2013) Machado and Figueiredo (2010)
↳ which are regular	polynomial	De Simone and Galluccio (2007)
line graphs	$\mathcal{NP}$-hard	Cai and Ellis (1991)
powers of cycles	polynomial	Meidanis (1998)
perfect graphs	$\mathcal{NP}$-hard	Cai and Ellis (1991)
↳ comparability graphs	$\mathcal{NP}$-hard	Cai and Ellis (1991)
↳ split-comparability graphs	polynomial	Sousa Cruz et al. (2017)
↳ which are d-regular for any constant $d \geqslant 3$	$\mathcal{NP}$-hard	Cai and Ellis (1991)
↳ cographs	open	
↳ quasi-thresholds	polynomial	Plantholt (1981)
↳ complete multipartite graphs	polynomial	Hoffman and Rodger (1992)
↳ chordal graphs	open	Figueiredo et al. (2000)
↳ indifference graphs	open	
↳ with odd Δ	polynomial	Figueiredo et al. (1997b)
↳ split-indifference	polynomial	Ortiz Z. et al. (1998)
↳ which are twin-free	polynomial	Figueiredo et al. (2003)

Table 1.1: Some results from the literature about the computational complexity of the problem of deciding the chromatic index of an n-vertex graph belonging to some restricted graph class (continued)

Graph class	Complexity	References
planar graphs	open	
↳ with $\Delta \leqslant 3$	polynomial	Tait (1880)
		Appel and Haken (1977)
		Appel et al. (1977)
↳ with $\Delta \geqslant 7$	polynomial	Sanders and Zhao (2001)
↳ with $\Delta \geqslant 8$	polynomial	Vizing (1965)
unichord-free graphs	$\mathcal{NP}$-hard	Machado et al. (2010)
↳ which are C_4-free	$\mathcal{NP}$-hard	Machado et al. (2010)
↳ with $\Delta \geqslant 4$	polynomial	Machado et al. (2010)
↳ which are C_6-free	$\mathcal{NP}$-hard	Machado et al. (2010)
↳ which are $\{C_4, C_6\}$-free	polynomial	Machado et al. (2010)

The proofs for all polynomiality results in Table 1.1 consist of characterising the chomatic index of the graphs in some given graph class in terms of some polynomial-time verifiable property, showing also a polynomial-time algorithm for Δ-edge-colouring the *Class 1* graphs of that class. This leads to a polynomial-time algorithm for optimal edge-colouring all graphs in these graph classes, since for the *Class 2* graphs one can use Vizing's algorithm. All constructive Δ-edge-colourability proofs in the works listed in Table 1.1 take advantage of the structural information of the graph classes dealt. It is interesting to observe that in graph classes with a little less structural information, the problem remains open, even with evidences for polynomiality and even in face of the combined effort of so many researches over the last decades.

In order to attack graph classes with not so much structural information, *recolouring* seems to be a convenient strategy. In fact, the recolouring procedures which we present allows us to construct Δ-edge-colourings even when we do not have much control over them. These procedures are *extensions* of Vizing's recolouring procedure in the sense that, when constructing a recolouring fan for an edge uv, if all vertices in the fan actually miss a colour, then our recolouring procedures behave like Vizing's. However, since we have only Δ colours available, the construction of the recolouring fan may be led to a neighbour of u which does not actually miss a colour (see Figure 1.15 on the next page). We show how to circumvent this problem if the graphs satisfy the conditions of our theorems.

Theorem 1.10 below is one of the main results which we have achieved with our recolouring procedures. In the statement, referring to the sum of the degrees of the neighbours of a vertex u as the *local degree sum* of u, we define the graph class $\mathscr{X}$, showing that every graph in $\mathscr{X}$ is *Class 1*.

Theorem 1.10. *Let $\mathscr{X}$ be the class of the graphs with maximum degree Δ whose majors have local degree sum bounded above by $\Delta^2 - \Delta$. All graphs in $\mathscr{X}$ are* Class 1.

Remark that only the local degree sums of the majors are relevant to bound above by $\Delta^2 - \Delta$, since the local degree sums of the non-majors already satisfy this bound, which implies that graphs with unitary core are trivially in $\mathscr{X}$.

As it is usual in the context of random graphs, we say that *almost every* graph, under some random graph model, satisfies some property $\mathscr{P}$ if the probability of a

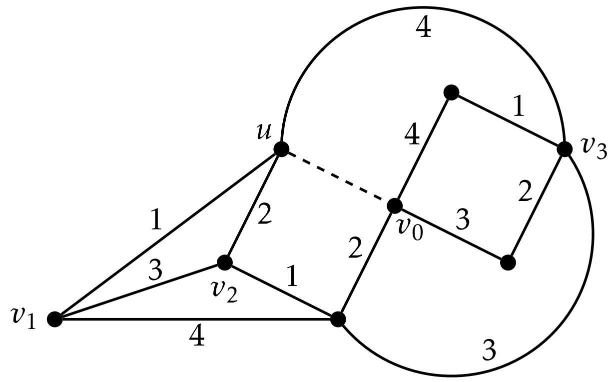


Figure 1.15: A Vizing's recolouring fan v_0, v_1, v_2, v_3 for an edge uv_0 in a graph G under a Δ-edge-colouring φ of $G - uv_0$. In this work, we refer to this graph G as the *crab claw* graph. Remark that the vertex v_0 misses $\varphi(uv_1)$, the vertex v_1 misses $\varphi(uv_2)$, the vertex v_2 misses $\varphi(uv_3)$, but there is no colour missing at v_3. Hence, despite the crab claw being a *Class 1* graph, we cannot continue to construct this fan.

random n-vertex graph[3] to satisfy $\mathscr{P}$ goes to 1 as n goes to ∞. In this work, we always consider the $\mathscr{G}(n, 1/2)$ random graph model.

Concerning Theorem 1.10, we further show that *almost every* graph is in $\mathscr{X}$, which means that $\mathscr{X}$ is a *large* graph class (see Figure 1.16). Actually, we show that

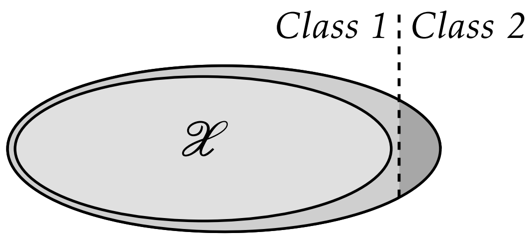


Figure 1.16: The graph class $\mathscr{X}$ and the set of all graphs partitioned according to Vizing's Theorem

almost every $\mathscr{G}(n, 1/2)$ graph is in $\mathscr{X}$ even given that the graph has cycles in the core. The reason why this is relevant is because the core of almost every graph is unitary (Erdős and Wilson, 1977), hence acyclic, and because every graph with acyclic core is already known to be *Class 1* (Fournier, 1977).

A *join graph* $G_1 * G_2$ is the result of the *join* of two disjoint graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, that is, the graph given by

$$V(G_1 * G_2) := V_1 \cup V_2 \qquad \text{and}$$
$$E(G_1 * G_2) := E_1 \cup E_2 \cup \{v_1 v_2 : v_1 \in V_1 \text{ and } v_2 \in V_2\}.$$

Figure 1.17 on the next page depicts two graphs and their join. Another example of a join graph is the diamond (Figure 1.2, p. 12), which is the $K_2 * \overline{K_2}$. Join graphs constitute an important graph class which appears in perfect graph theory (Lovász, 1972) and in efficient algorithms for many optimisation problems (Chvátal, 1975; Möhring, 1985). Moreover, a remarkable subclass of the join graphs are the (connected) P_4-free graphs (Corneil et al., 1981), mostly referred to as *cographs* in the literature. The computational

[3] We refer the reader to Bollobás (2001) for terms and concepts on random graphs. For instance, $\mathscr{G}(n, p)$ is the model on n vertices wherein, for every pair of distinct vertices u and v, the edge uv exists with probability p, being p in the closed real interval $[0, 1]$.

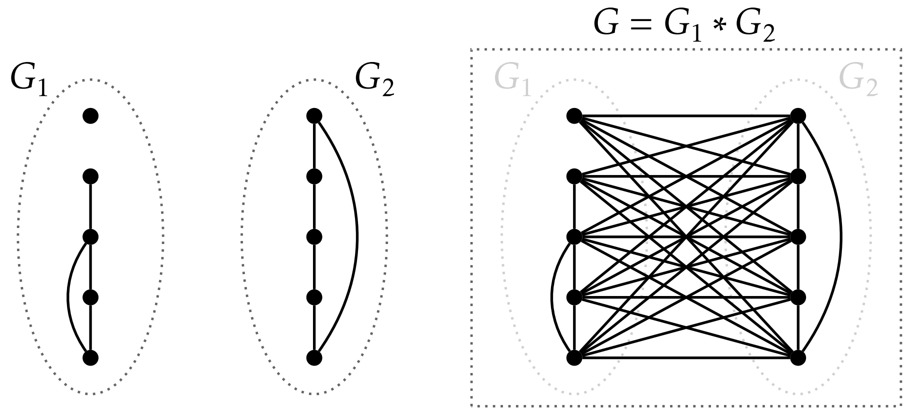


Figure 1.17: Two graphs, G_1 and G_2, and the join graph $G_1 * G_2$

complexity of determining the chromatic index of cographs is one of the open problems listed in the famous David Johnson's $\mathcal{NP}$-completeness column (Johnson, 1985). As shown by Corneil et al. (1985) and Alon and Stav (2008), almost every n-vertex graph can be turned into a cograph with no more than $\frac{1}{2}\binom{n}{2} - \frac{1}{5}\eta n^{\frac{1}{2}}$ edge addition or deletion operations (Corneil et al., 1985; Alon and Stav, 2008), being η a constant.

There has been a particular interest on edge-colouring join graphs in the last 15 years, because all n-vertex join graphs satisfy $\Delta \geqslant n/2$ and, as discussed in Chapter 2, a remarkable on edge-colouring suggests that the chromatic index can be decided in linear time for graphs with Δ this large (see Section 2.6 for more details). No polynomial-time algorithm is known for determining the chromatic index of a join graph yet, but some sufficient conditions for a join graph to be *Class 1* have been found. This work explores more of these conditions, as well as novel decomposition techniques for edge-colouring join graphs. In particular, we propose the following conjecture, for which we present some evidences.

Conjecture 1.11. *Let G_1 and G_2 be disjoint graphs such that $|V(G_1)| \leqslant |V(G_2)|$ without loss of generality. If $\Delta(G_1) \geqslant \Delta(G_2)$ and if the majors of G_1 induce an acyclic graph, then the join graph $G_1 * G_2$ is* Class 1.

Some of the evidences presented for Conjecture 1.11 have been found by developing another extended recolouring procedure.

Being G any graph with at least one vertex, the *prism GG* is the graph obtained from two copies of G by connecting with an edge each vertex in one copy to its corresponding vertex in the other. The *complementary prism $G\overline{G}$* is the graph obtained from the graphs G and $\overline{G}$ by connecting with an edge each vertex in G to its corresponding vertex in $\overline{G}$. Figure 1.18 on the next page depicts the prism K_3K_3, also referred to as the *triangular prism*, and the Petersen graph (Figure 1.7, p. 15) is actually the complementary prism $C_5\overline{C_5}$. It is easy to see that all prisms are *Class 1* and their edges can be optimally coloured in polynomial time, as the following observation clarifies:

Observation 1.12. *The prism GG is* Class 1 *for any graph G.*

Proof. Using Vizing's Theorem (Theorem 1.4), we construct the same $(\Delta(G)+1)$-edge-colouring for both copies of G. Now, for each vertex u in one of the copies, and being u'

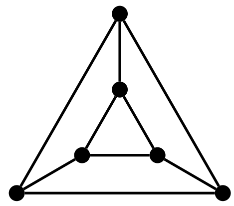


Figure 1.18: The prism graph $K_3 K_3$, often referred to as the *triangular prism*

the corresponding vertex in the other copy, both u and u' miss the same colours, so we can assign one of these colours to the edge uu'. □

We completely solve the edge-colouring problem for of *non-regular complementary prisms*, that is, we determine the chromatic index of the graphs in this graph class (see Theorem 1.13 below) and our proof is a polynomial-time algorithm to compute an optimal edge-colouring of any non-regular complementary prism.

Theorem 1.13. *A complementary prism can be* Class 2 *only if it is a regular graph distinct from the* K_2.

As argued by Haynes et al. (2009), complementary prisms constitute an important graph class, specially in the context of domination problems, as they generalise the concept of graph products and include important graphs, such as the Petersen graph. To the best of our knowledge, edge-colouring prisms and complementary prisms was not approached by any work in the literature.

This document is organised as follows:

- Chapter 2 discusses further results and theory developments found in the literature which are relevant for this work;
- Chapter 3 presents our first recolouring procedure and the proof for Theorem 1.10, as well as some other related results;
- Chapter 4 presents the evidences for Conjecture 1.11 and the proof for Theorem 1.13, as well as some other results on edge-colouring join graphs whose proofs are also constructive and yield polynomial-time procedures (not necessarily lying on recolouring);
- Chapter 5 presents some secondary results which we have found on the subject of total colouring during the development of our work on edge-colouring;
- Chapter 6 presents concluding remarks and discusses some challenges for future works.

Since we have avoided to concentrate all technical definitions in a single chapter or section, we have prepared a list of symbols in the beginning of the document and a short index in the end, in order to assist the reader in quickly recalling a specific term or notation usage. In the lexicographic sorting of the entries in the list of symbols and in the index, we have considered Greek letters as equivalent to their traditional transliterated counterparts in the Latin alphabet.

2 Further preliminaries

This chapter is organised as follows:

- Section 2.1 briefly presents a history of the origins of vertex and edge-colouring problems, relating it with the history of the *Four Colour Theorem*;
- Section 2.2 presents Vizing's Adjacency Lemma and other classical applications of Vizing's recolouring procedure in the literature;
- Section 2.3 discusses the relation of edge-colouring and matching and presents the classical result by Folkman and Fulkerson (1969) on the equitability property for edge-colourings;
- Section 2.4 briefly presents some computational complexity facts about edge-colouring;
- Section 2.5 defines some graph classes which are relevant for this work (in addition to the classes which have already been defined in Chapter 1) and also discusses some computational complexity aspects of edge-colouring problems when restricted to these graph classes;
- at last, Section 2.6 presents two major conjectures on edge-colouring which have been being subject of many works in the last 30 years, and which are closely related to the results of our own presented in the other chapters.

2.1 On the origins of main graph colouring problems

In 1852, the Irish mathematician William Hamilton received a letter from the British mathematician Augustus De Morgan, communicating a problem proposed by the South African mathematician Francis Guthrie. The problem came to De Morgan's knowledge by Guthrie's brother, a student of De Morgan. While trying to colour the map of England in a manner that no two adjacent regions were coloured the same, Guthrie realised that only four colours were necessary and conjectured that no map whatsoever would require more than this amount (see Figure 2.1 on the next page). Guthrie's conjecture (then called the *Four Colour Conjecture*, now *Four Colour Theorem*) remained open for over a century, inspiring the collaboration of several mathematicians, originating other problems in Combinatorics, and causing a deep development of many research branches. The first proof for the *Four Colour Theorem* was presented in 1977 (Appel and Haken, 1977; Appel et al., 1977) and was the first major proof in history to rely on the assistance of a computer, infeasible for a human to check by hand (Swart, 1980). A polynomial-time algorithm for 4-colouring any map was presented by Robertson et al. (1996). More on the history of the *Four Colour Theorem* can be found e.g. in Kubale (2004, Preface).

Regarding the regions of a map as *vertices* of a graph, whose edges are defined by the adjacencies between the corresponding regions, we reduce the problem of optimally

Figure 2.1: A 4-colouring of the map of Brazilian states using 4 shades of grey. Borders are depicted thick and white, which should not count as a colour.

colouring maps to the problem of computing an optimal *vertex*-colouring of a graph. A *k-vertex-colouring* (or simply *k-colouring*) of a graph G is an assignment $\varphi\colon V(G) \to \mathscr{C}$ of colours from a set $\mathscr{C}$ with $|\mathscr{C}| = k$ to the vertices of G in a manner that no two neighbours are assigned the same colour. The least k for which G is k-vertex-colourable is the *chromatic number* of G, denoted $\chi(G)$.

The corresponding graph of a map is clearly a *planar* graph, that is, a graph which can be *embedded* in the plane. We say that a graph can be *embedded* in a surface S if it can be drawn on S in a manner that two edges intersect if and only if they have a common end-vertex and only in that end-vertex. Since with a planar graph G one can also easily obtain at least one map to which G corresponds, the problems of optimally colouring maps and computing optimal vertex-colourings of planar graphs are equivalent. Therefore, the *Four Colour Theorem* is equivalent to the statement that the chromatic number of a planar graph is at most four.

The problem of deciding if a graph is 3-vertex-colourable is one of the 21 $\mathcal{NP}$-complete problems in the remarkable work by Karp (1972). This problem remains $\mathcal{NP}$-complete even when restricted to planar graphs Dailey (1980). Recall that deciding if a graph is 2-vertex-colourable in actually the problem of deciding if a graph is bipartite, which can be standardly solvable in linear time.

In the earliest studies on the *Four Colour Theorem* (*Conjecture*, by that time), mathematicians found some very peculiar graphs: the snarks[1]. With this discover lies perhaps the origin of edge-colouring, since a snark is defined as a non-3-edge-colourable connected cubic bridgeless graph. The relevance of the snarks to the *Four Colour Theorem* comes from the following result by Tait in 1880:

Theorem 2.1 (Tait, 1880). *The* Four Colour Theorem *is equivalent to the statement that no snark is planar.*

[1]Chladný and Škoviera (2010) discusses some peculiar and "somewhat mysterious" properties of the snarks.

The first snark to be found was the Petersen graph (Figure 1.7, p. 15), which, although credited to Petersen (1898), had already appeared in earlier works (Kempe, 1886). The proof for the non-3-edge-colourability of the Petersen graph is actually straightforward, once one knows the graph, and can be derived in many ways (Holton and Sheehan, 1992). It took almost 50 years for other snarks to be found, like the two 18-vertex snarks by Blanuša (1946 *apud* Costa et al., 2015) and the 210-vertex snark by Descartes[2] (1948). Snarks were so named by Gardner (1976) after the mysterious creature or thing hunted in the nonsense poem *The Hunting of the Snark*, by Lewis Carroll. The first infinite family of snarks was shown by Isaacs (1975). Later it was also shown that the number of snarks of any even order n is bounded below by $2^{(n-84)/18}$ (Skupień, 2007).

Observation 2.2 (Petersen, 1898). *The Petersen graph is not 3-edge-colourable.*

Proof (Naserasr and Škrekovski, 2003). If there is a 3-edge-colouring of the Petersen graph, then in such an edge-colouring no vertex misses a colour, and we shall prove that each one of the three colours appears at least twice in the outer C_5, a contradiction. Since the C_5 is not 2-edge-colourable, each colour α in the supposed 3-edge-colouring must appear at least once in the inner C_5. So, let uv be an α-coloured edge of the inner C_5, and let u' and v' be *the* neighbour of u and *the* neighbour of v, respectively, in the outer C_5. Recall that neither u' nor v' miss α. Hence, since u' and v' are not adjacent to each other, there must be at least two α-coloured edges in the outer C_5. □

Another relation between vertex and edge-colouring is the fact that any edge-colouring of a graph G can be viewed as a vertex-colouring of the *line graph* of G, and vice-versa. The *line graph* of G, denoted $\mathrm{L}(G)$, is the graph whose vertices are the edges of G and whose edges represent the adjacencies between the edges of G. Many results on edge-colouring in the literature are stated as results on vertex-colouring line graphs.

2.2 Other classical applications of Vizing's recolouring

We dedicate this section to some classical results from the literature which follow directly from Vizing's recolouring procedure, with no need of extending it.

Recall that Lemma 1.9 (p. 21) requires every neighbour of the vertex u (the vertex around which we are constructing a recolouring fan for an edge uv) to miss a colour, which of course always holds in a $(\Delta+1)$-edge-colouring. However, there are some situations in which we are constructing a Δ-edge-colouring and, because of the structure of the graph, we successfully manage to colour the edges in an appropriate order so that, at each edge uv considered, all neighbours of u miss a colour and thus Lemma 1.9 is satisfied. The proof for Theorem 2.3 below is an example of such an application of Vizing's recolouring procedure.

Theorem 2.3 (Fournier, 1977). *Every graph with acyclic core is* Class 1.

Proof. Let G be a graph with acyclic core and maximum degree $\Delta \geqslant 2$, since if $\Delta \leqslant 1$ then G is clearly *Class 1*. Since $\Lambda[G]$ is a forest, we choose for each component of $\Lambda[G]$ (a tree) an arbitrary vertex to be the *root* of the tree. For all $u \in V(\Lambda[G])$, let $h(u)$ and

[2]Blanche Descartes was a collaborative pseudonym used by R. L. Brooks, A. H. Stone, C. Smith, and W. T. Tutte. However, the article wherein the Descartes snark is shown is a work by Tutte solely.

$p(u)$ be the *height* and the *parent* of u in its tree, following the usual definitions in the context of rooted trees. In particular, $h(r) = 0$ and $p(r)$ is undefined for every root r. Now, we take $F := E(\Lambda[G])$ and, for each *trivial* tree in $\Lambda[G]$ (i.e. for each tree consisting of a single vertex u), we choose an arbitrary edge incident to u and add this edge to F. This way, we get $\Delta(G-F) < \Delta$, so we can take an Δ-edge-colouring of $G - F$ by Vizing's Theorem (see Figure 2.2).

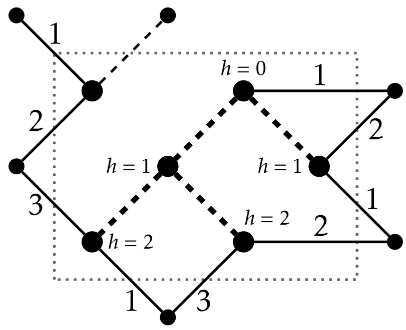


Figure 2.2: A $\Delta(G)$-edge-colouring of $G-F$ for a graph G and an edge set F as in the proof of Theorem 2.3. The edges of F are those which appear dashed since they have not been coloured yet, and thicker if they are also in $E(\Lambda[G])$. The vertices of $\Lambda[G]$, also thicker, are those within the dotted rectangle, followed by their heights if they belong to a non-trivial tree in $\Lambda[G]$.

Now we shall colour the edges in F. We first consider each edge uv such that v belongs to a unitary component of $\Lambda[G]$ and $u \notin V(\Lambda[G])$. Since v is the only major incident to u and the edge uv is not coloured yet, all neighbours of u miss at least one colour each from the colour set $\mathscr{C}$ used. Therefore, the edge uv satisfies the condition of Lemma 1.9 for Vizing's recolouring procedure and can be assigned a colour from $\mathscr{C}$.

It remains to colour the edges of $E(\Lambda[G])$. For each i from 1 to $\max_{u \in V(\Lambda[G])} h(u)$, we consider all vertices of $V(\Lambda[G])$ with height i, one at a time. For each vertex u considered, we shall colour the edge $up(u)$. For each neighbour w of u, we have one of the three following possibilities:

1. w is not a major of G, in which case w clearly misses a colour from $\mathscr{C}$;

2. w is a major of G with height greater than $h(u)$, in which case w misses a colour because the edge wu has not been coloured yet;

3. $w = p(u)$, in which case w also misses a colour because the edge $up(u)$ has not been coloured yet.

Ergo, the edge uv also satisfies the condition of Lemma 1.9. □

Besides acyclicness, further sufficient conditions related to the core of G for G to be *Class 1* have been established e.g. by Cariolaro and Cariolaro (2003), Akbari and Ghanbari (2012), and Akbari et al. (2012).

In the proof of Theorem 2.3, it should be noticed that, although we consider the edges of the core to be coloured one by one according to a *breadth-first* search order, a *depth-first* search order would also work[3]. Theorem 2.3 also follows from an earlier classical result, known as Vizing's Adjacency Lemma (Lemma 2.4), presented in 1965 by Vizing's second fundamental work on edge-colouring (Vizing, 1965). Vizing's

[3]We refer the reader to Diestel (2010) for more on graph search algorithms.

Adjacency Lemma is much explored in several works on edge-colouring, e.g. Chetwynd and Hilton (1985, 1986); Hilton and Cheng (1992). In the statement of the lemma, a *critical* graph is a connected *Class 2* graph G with $\chi'(G-e) < \chi'(G)$ for all $e \in E(G)$. It is clear that every *Class 2* graph G has a critical Δ-subgraph[4].

Lemma 2.4 (Vizing's Adjacency Lemma (Vizing, 1965 *apud* Stiebitz et al., 2012)). *For every edge uv of a critical graph G, the number of majors adjacent to u in G, being $\Delta := \Delta(G)$, is at least:*

$$\begin{aligned} \Delta - d_G(v) + 1\,, &\quad \textit{if } d_G(v) < \Delta; \\ 2\,, &\quad \textit{if } d_G(v) = \Delta. \end{aligned}$$

Proof. Consider the following:

Claim. Let uv be an edge of a graph G such that $\chi'(G-uv) = \Delta(G-uv) = \Delta(G) =: \Delta$. If u is adjacent in $G-uv$ to at most $\Delta - d_G(v)$ majors of G, then G is *Class 1*.

We show how the lemma straightforwardly follows from the claim. Let G be a critical graph and let uv be any edge of G. We know that $\chi'(G-uv) \leqslant \Delta$ by the criticality of G. We also know that $\Delta(G-uv) = \Delta(G)$ because, otherwise, every vertex of $G-uv$ would miss a colour in any $\Delta(G)$-edge-colouring and so we would be able to construct a $\Delta(G)$-edge-colouring for G by Lemma 1.9. For the sake of contradiction, we assume that the number of majors of G adjacent to u in G is at most:

$$\begin{aligned} \Delta - d_G(v)\,, &\quad \text{in the case wherein } d_G(v) < \Delta; \\ 1\,, &\quad \text{in the case wherein } d_G(v) = \Delta. \end{aligned}$$

Remark in the case wherein $d_G(v) = \Delta$ that the assumption implies that v is the only major adjacent to u in G. Remark also that in both cases we have u adjacent in $G-uv$ to at most $\Delta - d_G(v)$ majors of G. Hence, by the claim, G is *Class 1*, contradicting the fact that critical graphs are *Class 2* by definition.

Now we shall prove the claim. We start with an edge-colouring φ of $G-uv$ using a colour set $\mathscr{C}$ with Δ colours. In view of Vizing's recolouring procedure (Lemmas 1.6–1.9, p. 20–21), we shall colour the edge uv with one of the colours of $\mathscr{C}$, possibly causing the recolouring of some of the other edges of G.

If $d_G(v) = \Delta$, then u is adjacent in G to no other major of G besides v by hypothesis. Since the edge uv has not been coloured yet, every neighbour of u in G misses at least one colour from $\mathscr{C}$ each, so we can apply Lemma 1.9 and we are done.

If $d_G(v) < \Delta$, then the numbers of colours of $\mathscr{C}$ missing at u is at least one, and at $v =: v_0$ is exactly $\Delta - d_G(v) + 1 \geqslant 2$. If there is a recolouring fan $v_0, \ldots, v_k$ for uv such that v_k misses a colour missing at u or a colour missing at v_j for some $j < k$, we are done by Lemma 1.6 or Lemma 1.8, respectively. Otherwise, for every maximal recolouring fan $v_0, \ldots, v_k$ for uv, the vertex v_k is a major of G and hence misses no colour from $\mathscr{C}$.

Since for each one of the $\Delta - d_G(v) + 1 \geqslant 2$ colours missing at v_0 we have at least one distinct maximal recolouring fan for uv starting at v_0 and ending at a major of G, and since u is adjacent to at most $\Delta - d_G(v)$ majors of G, by the Pigeonhole Principle there must be two distinct maximal recolouring fans $v_0, w_1, \ldots, w_\ell$ and $v_0, x_1, \ldots, x_r$ such

[4]Actually, in the same work Vizing also showed that if G is a *Class 2* graph, then for every $k \in \{2, \ldots, \Delta\}$ the graph G has a critical subgraph H with $\Delta(H) = k$.

that $w_i = x_j$ for some $i \in \{2,\ldots,\ell\}$ and some $j \in \{2,\ldots,r\}$. We assume without loss of generality that $w_{i'} \neq x_{j'}$ for all $i' \in \{1,\ldots,i-1\}$ and all $j' \in \{1,\ldots,j-1\}$.

Let α be the colour of the edge $uw_i = ux_j$, let β be a colour missing at u, and let P be the α/β-path to which u belongs. Since w_{i-1} and x_{j-1} are distinct vertices which miss α, but at most one of them can be the other outer vertex of P, one of them, say w_{i-1}, is surely not in P. Then, by exchanging the colours along the α/β-path to which w_{i-1} belongs, we obtain β missing at both u and w_{i-1}. Ergo, $v_0, w_1, \ldots, w_{i-1}$ is now a Vizing's complete recolouring fan, and applying Lemma 1.6 concludes the proof. □

From Vizing's Adjacency Lemma follow the facts listed in Theorem 2.5.

Theorem 2.5 (Vizing, 1965 *apud* Stiebitz et al., 2012). *Let G be any* Class 2 *graph with maximum degree Δ. Then:*

(i) *the number of majors of G is at least*

$$\max(\{3\} \cup \{\Delta - \delta(H) + 2 : H \textit{ critical } \Delta\textit{-subgraph of } G\}).$$

(ii) *the number of edges of G is at least* $(3\Delta^2 + 6\Delta - 1)/8$.

Proof. Since G has neither more edges nor more majors than its critical Δ-subgraphs, it suffices to show that if H is a critical Δ-subgraph of G with minimum degree δ, then H has at least $\max\{3, \Delta - \delta + 2\}$ majors of H (which are also majors of G) and at least $(3\Delta^2 + 6\Delta - 1)/8$ edges.

According to Lemma 2.4, every vertex u of H must be adjacent in H to at least two majors, even if u is itself a major, so we have at least three majors in H.

Now we shall prove that H has at least $\Delta - \delta + 2$ majors. Let u be a vertex of degree δ in H. By Lemma 2.4, we know that u is adjacent in H to at least two majors, so let v be one of these majors. We also know, again by Lemma 2.4, that v is adjacent in H to at least $\Delta - \delta + 1$ majors. Therefore, H has at least $\Delta - \delta + 2$ majors.

It remains to demonstrate (ii). By the fact that H has at least $\Delta + 1$ vertices, amongst which at least $\Delta - \delta + 2$ are majors and the others have degree at least δ in H, follows that

$$2|E(H)| = \sum_{u \in V(H)} d_H(u) \geqslant (\Delta - \delta + 2)\Delta + (\delta - 1)\delta,$$

hence

$$|E(H)| \geqslant \frac{\delta^2 - (\Delta + 1)\delta + \Delta^2 + 2\Delta}{2}.$$

Since by differentiation we get

$$\frac{\mathrm{d}}{\mathrm{d}\delta}\left(\frac{1}{2}\delta^2 - \frac{1}{2}(\Delta + 1)\delta + \frac{1}{2}(\Delta^2 + 2\Delta)\right) = \delta - \frac{1}{2}(\Delta + 1),$$

the function $f(\delta) := (\delta^2 - (\Delta + 1)\delta + \Delta^2 + 2\Delta)/2$ reaches its minimum when $\delta = (\Delta + 1)/2$. Ergo,

$$|E(H)| \geqslant f\left(\frac{\Delta + 1}{2}\right) = \frac{3\Delta^2 + 6\Delta - 1}{8}.$$ □

Now we show how Theorem 2.3 straightforwardly follows from Vizing's Adjacency Lemma (Lemma 2.4).

Alternative proof for Theorem 2.3. Let G be a graph with acyclic core and assume, for the sake of contradiction, that G is *Class 2*. Then G clearly has a critical Δ-subgraph H whose core is also acyclic. Since acyclic graphs must have vertices of degree less than two, all majors of H which have degree one in its core contradict Lemma 2.4. $\square$

We close this section with a result which implies that the problem of constructing an optimal edge-colouring of a graph G can be reduced to the problem of constructing an optimal edge-colouring of the *semi-core* of G, that is, the subgraph (which we denote $\cancel{\mathrm{A}}[G]$) of G induced by its majors and the neighbours of its majors. From the proof follows that, once we have an optimal edge-colouring of the *semi-core* of G, the edges of G not in its semi-core can be coloured in polynomial time with Vizing's recolouring procedure.

Theorem 2.6 (Machado and Figueiredo, 2010). *The chromatic index of any graph is equal to the chromatic index of its semi-core.*

Proof. Observing that $\Delta(\cancel{\mathrm{A}}[G]) = \Delta(G)$, let φ be an optimal edge-colouring of $\cancel{\mathrm{A}}[G]$, being $\mathscr{C}$ the colour set used. Now, we shall colour the edges of $E(G) \setminus E(\cancel{\mathrm{A}}[G])$, one at a time. For each edge uv considered, at least one end-vertex of uv, say u, is not in $V(\cancel{\mathrm{A}}[G])$. Hence, all neighbours of u in G are not majors of G, so they all miss at least one colour from $\mathscr{C}$ each. The proof is therefore concluded by Lemma 1.9. $\square$

2.3 Edge-colouring, matchings, and equitability

A *matching* in a graph G is a set $M \subseteq E(G)$ whose edges are all pairwise non-adjacent. If the set of the vertices of G *covered* by M (i.e. the set of the ends of the edges in M) equals $V(G)$, then M is said to be *perfect* in G. If M covers all vertices of G but one, then M is said to be *near-perfect*. The cardinality of the maximum matching in G, also referred to as the *matching number* of G, is denoted $\nu(G)$ in this work. By the way, amongst other graph parameters which appear in this document are the *independent set number* (i.e. the size of the maximum independent set in G), denoted $\alpha(G)$, and the *clique number* (i.e. the size of the maximum clique in G), denoted $\omega(G)$, of G.

Berge's Lemma, transcribed below, is a classical argument which appears in many results on matchings in graphs. In the statement of the lemma, an M-augmenting path in a graph G is a path in G whose edges alternate between edges in and not in M, and whose outer vertices are not covered by M.

Lemma 2.7 (Berge's Lemma (Berge, 1957)). *A matching M in a graph G is maximum if and only if there is no M-augmenting path in G.*

Proof. If there is an M-augmenting path in G, then M is surely not maximum, since by taking $M' := (M \cup E(P)) \setminus (M \cap E(P))$ we obtain a matching larger than M. Therefore, it remains to prove that if there is no M-augmenting path in G, then M is maximum.

We shall prove the contraposition of the aimed inference. Let M be a non-maximum matching in G. We shall prove that there is an M-augmenting path in G. Let W be a maximum matching in G, and let $M' := M \setminus W$ and $W' := W \setminus M$. Because M' and W' are disjoint, each component of $G[M' \cup W']$ is clearly a path or an even cycle whose edges alternate between edges in M' and in W'. Furthermore, since $|W'| > |M'|$, at least one of the components is an odd-length path whose outer vertices are not covered by M', that is, an M-augmenting path. $\square$

Let $\varphi\colon E(G) \to \mathscr{C}$ be an edge-colouring of a graph G and let $\alpha \in \mathscr{C}$. It follows by definition that the set of the edges coloured α is a *matching*. Therefore, a *k-edge-colouring* of a graph G can be equivalently defined as a partition of $E(G)$ in k (disjoint and non-empty[5]) matchings. Since a matching can contain at most $\lfloor n/2 \rfloor$ edges, this provides another proof for Observation 1.1 (p. 16). Recall that not only the set of the edges coloured α is a matching, for any colour $\alpha \in \mathscr{C}$, but also that for any two colours α and β in $\mathscr{C}$, the subgraph induced by the edges coloured α or β is a disjoint union of paths and even cycles. This fact brings the following powerful result on edge-colouring. In the statement, an edge-colouring is said to be *equitable* if, for all two colours α and β used in the colouring, the number of edges coloured α differs from the number of edges coloured β by at most one.

Theorem 2.8 (Folkman and Fulkerson, 1969). *If a graph G admits a k-edge-colouring for some integer k, then G also admits an equitable k-edge-colouring.*

Proof. Let $\varphi\colon E(G) \to \mathscr{C}$ be an non-equitable k-edge-colouring of G and, for all $\alpha \in \mathscr{C}$, let m_α denoted the number of edges coloured α. We shall demonstrate how to decrease by one the difference between m_α and m_β for every pair of colours α and β in $\mathscr{C}$ such that $|m_\alpha - m_\beta| \geqslant 2$. Then, we shall be able to obtain an equitable k-edge-colouring for G by repeatedly applying this procedure.

Assume without loss of generality that $m_\alpha > m_\beta$. This can only be possible if one of the components of $G_\varphi[\alpha, \beta]$ is an odd-length path P with α-coloured edges incident to its outer vertices, since in an even-length path or in an even cycle the number of edges coloured α equals the number of edges coloured β. Hence, by exchanging the colours along P, we simultaneously decrease m_α and increase m_β by one. $\square$

It is interesting to comment that the corresponding result on vertex-colourings also holds, but only for $k \geqslant \Delta{+}1$ (Hajnal and Szemerédi, 1970). This was first conjectured by Paul Erdős in 1964, generalising an earlier result by Corrádi and Hajnal (1963). The proof, far less simple than the proof for Theorem 2.8, came in 1970 by Hajnal and Szemerédi, and the result is now often referred to as the Hajnal–Szemerédi Theorem.

Given that Theorem 2.8 establishes that every graph admits an equitable k-edge-colouring for every $k \geqslant \chi'(G)$, one may wonder how *inequitable* a k-edge-colouring can be and if any graph admits a k-edge-colouring satisfying the property chosen for the *inequitability* criterion. No matter which property we choose, we already know that no colour in an edge-colouring of a graph G can be assigned to more than $\nu(G)$ edges. Hence, it is relevant to ask whether any k-edge-colourable graph admits a k-edge-colouring wherein there is a colour assigned to exactly $\nu(G)$ edges. The answer is affirmative when $k \geqslant \Delta(G) + 1$. This and other interesting remarks constitute Observation 2.9 below. The *net* (Figure 2.3) is an example of a *Class 1* graph for which no colour in a Δ-edge-colouring can be assigned to $\nu(G)$ edges.

Observation 2.9. *Let G be a graph of maximum degree Δ and let k be a positive integer. All of the following hold:*

(i) *if G is a* Class 1 *graph, then it has a maximum matching which covers all its majors;*

[5]Recall that disjointness and non-emptiness of the parts follow from the very definition of a partition.

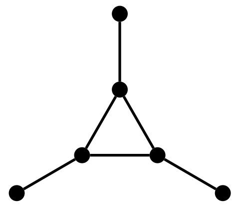


Figure 2.3: The *Class 1* complementary prism $K_3\overline{K_3}$, often referred to as the *net*. Since the removal of its unique perfect matching decreases its maximum degree but not its chromatic index, no colour in a Δ-edge-colouring of this graph can be assigned to three edges.

(ii) *if* $k \geqslant \Delta + 1$, *then* G *admits a* k*-edge-colouring in which one of the colours is assigned to* $\nu(G)$ *edges.*

Proof. (i) Let G be a *Class 1* graph and let M be a matching covering all majors of G with the largest number of edges amongst all matchings which cover all majors of G. The existence of M follows from the fact that, for every colour $\alpha \in \mathscr{C}$ in a Δ-edge-colouring $\varphi\colon E(G) \to \mathscr{C}$, the set of the edges coloured α is a matching which covers all majors of G, since no major misses a colour of $\mathscr{C}$. For the sake of contradiction, assume that M is not a maximum matching, which by Berge's Lemma (Lemma 2.7, p. 35) implies the existence of an M-augmenting path P in G. However, by taking $M' := (M \cup E(P)) \setminus (M \cap E(P))$, we obtain a matching which still covers all majors of G, but larger than M, contradicting the assumption on the cardinality of M.

(ii) In order to prove the statement for $k = \Delta + 1$, which suffices, we claim that every graph G has a maximum matching M such that $\chi'(G - M) \leqslant \Delta$. If this claim holds, then it is immediate to construct a $(\Delta + 1)$-edge-colouring of G reserving one of the colours to the matching M. In order to prove the claim, take any maximum matching M in G. If M covers all majors of G, then $\Delta(G - M) \leqslant \Delta - 1$, so $\chi'(G - M) \leqslant \Delta$ by Vizing's Theorem (Theorem 1.4, p. 19). On the other hand, if the set U of the majors of G which are not covered by M is not empty, then U must be an independent set, otherwise an edge uv with both u and v in U would bring a matching $M + uv$ larger than M. Therefore, $G - M$ is a graph with $\Delta(G - M) = \Delta(G)$ whose core is $G[U]$ and hence edgeless. Ergo, $G - M$ is a *Class 1* graph by Vizing's Adjacency Lemma (Lemma 2.4). □

From Observation 2.9 follows an important remark on our result on edge-colouring complementary prisms stated in Theorem 1.13 (p. 27). Although we show that a non-regular complementary prism $G\overline{G}$ is *Class 1*, this does not mean that $G\overline{G}$ admits a $\Delta(G\overline{G})$-edge-colouring wherein one of the colours is assigned to the perfect matching between G and $\overline{G}$ (recall Figure 2.3, p. 36). This means that our proof cannot rely on the strategy of reserving a colour for the perfect matching between G and $\overline{G}$ and then try to colour G and $\overline{G}$ with $\Delta(G\overline{G}) - 1 = \max\{\Delta(G), \Delta(\overline{G})\}$ colours, since this shall fail whenever the graph with higher maximum degree is a *Class 2* graph.

2.4 On the hardness of edge-colouring

Let CHRIND be the decision problem defined by:

CHRIND:

Instance: a graph G;

Question: Is G *Class 1*?

This problem is clearly in $\mathcal{NP}$, that is, each positive instance (a *Class 1* graph) for the problem has a polynomial-time verifiable polynomial-length certificate (a Δ-edge-colouring)[6]. The first proof of the $\mathcal{NP}$-completeness of CHRIND was presented by Holyer (1981), which reduced the classical $\mathcal{NP}$-complete problem 3SAT to CHRIND. Since the reduction shown outputs a cubic graph for every 3SAT formula given, the restriction of CHRIND to cubic graphs is also an $\mathcal{NP}$-complete problem.

The $\mathcal{NP}$-completeness of CHRIND relates the problem to one of the most important open question in Computer Science: *Is $\mathcal{P}$ (the class of the decision problems which can be decided in polynomial time) equal to $\mathcal{NP}$ (the class of the decision problems whose positive instances have polynomial-length certificates which can be verified in polynomial time)?*[7] This is because the existence of a polynomial-time algorithm for any $\mathcal{NP}$-complete problem would imply $\mathcal{P} = \mathcal{NP}$. Since it is widely conjectured that $\mathcal{P} \neq \mathcal{NP}$ (Aaronson, 2017), a polynomial-time algorithm for CHRIND (and hence for the problem of actually optimal edge-colouring graphs) is unlikely to exist.

Another major open question in Computer Science is: *Is $\mathcal{NP}$ equal to* co$\mathcal{NP}$*?* The computational complexity class co$\mathcal{NP}$ consists of the complements of the problems in $\mathcal{NP}$. This means that the problems in co$\mathcal{NP}$ are those whose *negative* instances have polynomial-length certificates (in this context often referred to as *disqualifiers*) which can be verified in polynomial time. By the definitions of the classes follows that $\mathcal{P} \subseteq \mathcal{NP} \cap \mathrm{co}\mathcal{NP}$ (see Figure 2.4).

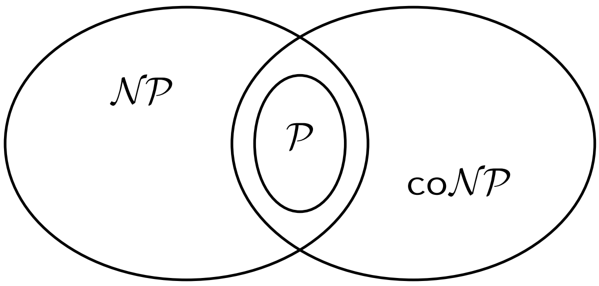


Figure 2.4: The known relations amongst the computational complexity classes $\mathcal{P}$, $\mathcal{NP}$, and co$\mathcal{NP}$

We know that $\mathcal{P} = \mathcal{NP}$ implies $\mathcal{NP} = \mathrm{co}\mathcal{NP}$, but it is still open if the inference also holds in the converse direction (Sipser, 1992; Papadimitriou, 1994). Since it is widely believed that $\mathcal{NP} \neq \mathrm{co}\mathcal{NP}$ (Ibid.), and since the complement of CHRIND (i.e. the problem of deciding if a graph is *Class 2*) is an co$\mathcal{NP}$-complete problem, it is unlikely that all *Class 2* graphs admit polynomial-time verifiable polynomial-length certificates. This can be interpreted as follows: differently from a *Class 1* graph (which can be proved to be *Class 1* by showing one of its Δ-edge-colourings), no method for proving that a graph is *Class 2* is known which works for all *Class 2* graphs and which yields polynomial-time verifiable polynomial-length proofs. If such a method were found, then it would imply $\mathcal{NP} = \mathrm{co}\mathcal{NP}$.

As we have mentioned, Holyer (1981) showed that CHRIND is $\mathcal{NP}$-complete when restricted to cubic graphs, which implies the $\mathcal{NP}$-completeness of CHRIND for general graphs. However, it could be the case that CHRIND is $\mathcal{NP}$-complete only because it is $\mathcal{NP}$-complete for cubic graphs, being polynomial for d-regular graphs with $d > 3$

[6]We refer the reader to Papadimitriou (1994) for terms and concepts on computational complexity.

[7]The question *Is $\mathcal{P}$ equal to $\mathcal{NP}$?* is one of the seven still open *Millennium Prize Problems*, a list of (initially eight) problems stated by the Clay Mathematics Institute (Jaffe, 2000). The institute offered one million dollars for each problem, to be awarded for those who present a solution for the problem.

(recall that for $d < 3$ the problem is in fact polynomial). But this is *not* the case. For every constant $d \geqslant 3$, the restriction of CHRIND to d-regular graphs remains $\mathcal{NP}$-complete (Leven and Galil, 1983). On the other hand, if we consider a graph class wherein the maximum degree $\Delta(G)$ is not a constant, but bounded from below by any $\Omega(|V(G)|)$ function (in particular $\varepsilon|V(G)|$, being $\varepsilon > 0$ a constant), CHRIND seems to be a less hard problem (see Observation 2.10 below) and it is actually conjectured to be in $\mathcal{P}$ for $\Delta(G) > n/3$ (see Section 2.6).

Observation 2.10 (inspired in Galby et al. (2018)). *Being $k \in \mathbb{N}$ a fixed constant, the problem*

k-CHRIND:

Instance: *a graph G;*
Question: *Is G k-edge-colourable?*

is polynomial when restricted to graphs G with $\Delta(G) \geqslant f(|V(G)|)$, for any $\Omega(|V(G)|)$ function $f : \mathbb{N} \to \mathbb{R}$.

Proof. Our polynomial algorithm works on a given input graph G as follows:

1. First, the algorithm finds in linear time the maximum degree Δ of G.
2. If $\Delta \neq k$, the algorithm outputs "yes" if $\Delta < k$ or "no" if $\Delta > k$, in view of Vizing's Theorem (Theorem 1.4).
3. If $\Delta = k$, then we know that the number n of vertices of G and the number m of edges of G are at most $k/f(n)$ and $(k/f(n))^2$, respectively. Since $1/f(n) = O(1)$, we have that in this case $k/f(n)$ and $(k/f(n))^2$ are both bounded above by a constant which does not depend on the size of the input graph. Therefore, any brute-force search can determine the k-edge-colourability of G in constant time. For instance, we can use the set partition algorithm by Björklund et al. (2009), which uses the principle of inclusion–exclusion and, when applied to connected graphs with m edges, yields an $O(2^m m^{O(1)})$-time exact edge-colouring algorithm. Remark that this is a constant time when m is bounded above by a constant. □

Observation 2.11 (Galby et al., 2018). *The restrictions of k-CHRIND to cographs and to join graphs are polynomial.*

Proof. Follows from Observation 2.10 and from the facts that: n-vertex join graphs have maximum degree $\Delta \geqslant n/2$; every connected component of a cograph is a join graph (Corneil et al., 1981); the chromatic index of any graph is the maximum amongst the chromatic indices of its connected components. □

As listed in Table 1.1 (p. 22), Cai and Ellis (1991) showed the $\mathcal{NP}$-completeness of CHRIND when restricted to several graph classes, including the d-regular graphs with girth at least k for any constants $d, k \geqslant 3$ (the special case of CHRIND restricted to triangle-free cubic graphs was also proved to be $\mathcal{NP}$-complete by Koreas (1997), probably without knowing the earlier work by Cai and Ellis (1991)). This implies that, for every graph H containing a cycle, the restriction of CHRIND to H-free graphs is $\mathcal{NP}$-complete. In the same work the authors showed that the restriction of CHRIND to $K_{1,3}$-free graphs is also $\mathcal{NP}$-complete (the complete bipartite graph $K_{1,3}$ is often referred to as the *claw*, see Figure 2.5). This brings the following:

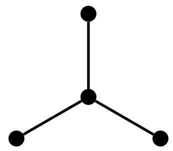


Figure 2.5: The $K_{1,3}$, also known as the *claw*

Observation 2.12. *Let H be a graph. Then, the restriction of* CHRIND *to H-free graphs is $\mathcal{NP}$-complete if at least one of the components of H is not a path.*

Remark that Observation 2.12 fails to provide a full polynomial dichotomy of CHRIND for H-free graphs because the problem is still open in the case wherein H is *linear forest* (i.e. a disjoint union of paths) with at least one of its components being a P_k for $k \geqslant 4$. Recall that the P_4-free graphs are the cographs. Recall also that the P_3-free graphs are the disjoint unions of complete graphs, for which CHRIND is already solved (see Theorem 1.2, p. 17). On the other hand, considering now the k-CHRIND problem, a full polynomial dichotomy of the problem for H-free graphs was recently established:

Theorem 2.13 (Galby et al., 2018). *Let k be a positive integer and H be a graph. Then, the restriction of k-*CHRIND *to H-free graphs is:*

(i) *$\mathcal{NP}$-complete, if at least one of the components of H is not a path;*

(ii) *polynomial, otherwise (that is, if H is a linear forest).*

We close this section highlighting that, differently from CHRIND, a polynomial dichotomy for the problem defined by

CHRNUM :

Instance: a graph G and a positive integer k;
Question: Is G k-vertex-colourable?

has already been established for H-free graphs:

Theorem 2.14 (Král et al., 2001). *Let H be a graph. Then, the restriction of* CHRNUM *to H-free graphs is:*

(i) *polynomial, if H is an induced subgraph of the P_4 or of the $P_3 \cup K_1$;*

(ii) *$\mathcal{NP}$-complete, otherwise.*

2.5 On edge-colouring some other graph classes

We have mentioned in Table 1.1 (p. 22) the results from the literature about the computational complexity of the CHRIND problem when restricted to some graph classes. Amongst the graph classes which appear in the table, some have already been defined in Chapter 1. In this section, we define a few more graph classes which are relevant for this work, highlighting some computational complexity aspects of edge-colouring restricted to these graph classes. For the graph classes which appear in Table 1.1 but are not defined in any chapter of this text, information can be found in the references given in the table.

An important graph class for the history of graph colouring problems is the class of the *perfect graphs*. A graph G is said to be *perfect* if, for every induced subgraph H of G, the chromatic number of H equals the size of the maximum clique in H. As shown for example by Grötschel et al. (1988, Chapter 9), many problems which are $\mathcal{NP}$-hard for graphs in general are polynomially solvable for perfect graphs using linear programming, such as computing an optimal vertex-colouring and finding a maximum clique or a maximum independent set. A remarkable result on perfect graphs is the *Strong Perfect Graph Theorem* (Theorem 2.15), which was first conjectured by Berge in 1961 and took over 40 years to be proved.

Theorem 2.15 (Strong Perfect Graph Theorem (Chudnovsky et al., 2006)). *A graph G is perfect if and only if neither G nor $\overline{G}$ has an induced odd cycle of length at least five.*

A polynomial-time recognition algorithm for perfect graphs was also shown by Chudnovsky et al. (2005).

Although so many hard problems can be efficiently solvable for perfect graphs, the CHRIND problem remains $\mathcal{NP}$-complete for the class even when restricted to comparability graphs, a subclass of perfect graphs (Cai and Ellis, 1991).

Other interesting subclasses of perfect graphs are the cographs (Corneil et al., 1981) and the chordal graphs (Fulkerson and Gross, 1965). A graph is said to be *chordal* if it has no induced cycle with length greater than three. The diamond (Figure 1.2, p. 12) and the Hajós graph (Figure 2.6) are examples of chordal graphs. A well-known

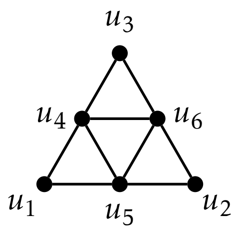


Figure 2.6: A perfect elimination order $u_1,\dots,u_6$ of the Hajós graph

result on chordal graphs is that they can be characterised by a *perfect elimination order* (Fulkerson and Gross, 1965), as stated in Theorem 2.16. In the statement, a vertex u is said to be *simplicial* in a graph G if $\{u\} \cup N_G(u)$ induces a complete graph.

Theorem 2.16 (Fulkerson and Gross, 1965). *An n-vertex graph is chordal if and only if its vertices admit a* perfect elimination order $u_1,\dots,u_n$, *that is, an order such that, for every $i \in \{1,\dots,n\}$, the vertex u_i is simplicial in $G[\{u_i,\dots,u_n\}]$.*

Chordal graphs can be recognised in linear time (Rose et al., 1976), and the proof for this relies on a special graph search algorithm which can be used to output a perfect elimination order if the input graph is chordal. This special graph search algorithm is referred to as the *lexicographic breadth-first search* (shortly, *LexBFS*) in the literature. Restricted to chordal graphs, the computational complexity of CHRIND is still open, with some partial results achieved, as listed in Table 1.1.

A *circular-arc graph* G is the intersection graph of a finite set S of arcs of a circle, in which case S is an *arc model* corresponding to G. If there is an arc model corresponding to G wherein no arc properly contains another, then G is said to be a *proper* circular-arc graph. If there is an arc model corresponding to G wherein all the arcs have the same length, then G is said to be a *unit* circular-arc graphs. Homonymous terms are defined for *interval graphs* analogously, but being the *interval model S* a finite set of intervals on the real line, instead of a set of arcs of a circle. The classes of the unit interval graphs and of the proper interval graphs are the same Roberts (1969), and these graphs are often referred to as the *indifference* graphs in the literature. However, the classes of the unit circular-arc graphs and of the proper circular-arc graphs are not the same (see Figure 2.7).

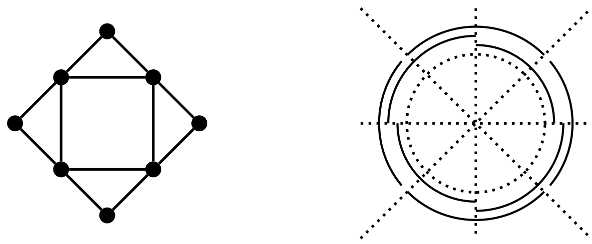


Figure 2.7: A proper non-unit circular-arc graph with a corresponding arc model. Remark that this graph is not chordal.

Interval and circular-arc graphs can be recognised in linear time (Booth and Lueker, 1976; McConnell, 2003). All interval graphs are chordal (Lekkerkerker and Boland, 1962) and clearly circular-arc graphs, but there are circular-arc graphs which are not chordal (the graph of Figure 2.7 is an example of a circular-arc graph which is not chordal). Interestingly, it can be straightforwardly demonstrated that the classes of chordal proper circular-arc graphs and of chordal unit circular-arc graphs are the same.

The vertices of an indifference graph admit an *indifference order*, that is, a linear order in which vertices belonging to the same maximal clique appear consecutively Looges and Olanu (1993). Analogously, the vertices of a proper circular-arc graph G admit a *proper circular-arc order*, that is, a circular order σ in which, for every edge $\overrightarrow{uv}$ under the clockwise orientation of the edges along σ, all the vertices clockwise between u and v induce a complete graph in G.

Although optimal vertex-colourings for indifference, and even for chordal, graphs can be computed in linear time (Rose et al., 1976), the computational complexity of determining the chromatic index of indifference graphs remains open, as discussed in Section 2.6. On the other hand, computing an optimal vertex-colouring for a circular-arc graph is $\mathcal{NP}$-hard (Garey et al., 1980), but polynomial when restricted to proper circular-arc graphs (Orlin et al., 1981). To the best of our knowledge, there was no published work on total or edge-colouring circular-arc graphs in the literature before the results presented in Chapter 5.

2.6 The 1-Factorisation and the Overfull Conjectures

An interesting fact which follows from Observation 1.1 (p. 16) is that graphs with more than $\Delta\lfloor n/2\rfloor$ edges are clearly *Class 2*. This argument has already been used in Chapter 1 to prove that odd cycles and complete graphs of odd order are *Class 2*. Graphs with more than $\Delta\lfloor n/2\rfloor$ edges are said to be *overfull*, and were so named by Chetwynd and Hilton (1984a), but the concept was already present in earlier works (e.g. Beineke and Wilson (1973)). A graph G is said to be *subgraph-overfull* (shortly, *SO*) if it has an overfull Δ-subgraph H (see Figure 2.8).

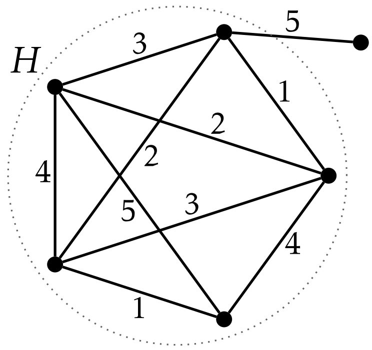


Figure 2.8: An overfull Δ-subgraph H of a non-overfull graph G and a $(\Delta+1)$-edge-colouring of G

It should be noticed that only odd-order graphs can be overfull (but an even-order graph can be *SO*, as the graph of Figure 2.8), since even-order graphs have at most $\Delta n/2$ edges. In fact, from the definition follows that all regular graphs of odd order are overfull. Moreover, overfull graphs can be equivalently defined as odd-order graphs which are regular or *very close* to be regular, as precisely specified in the following observation:

Observation 2.17 (Niessen, 1994). *An n-vertex graph G is overfull if and only if n is odd and $\sum_{u\in V(G)}(\Delta(G)-d_G(u)) \leqslant \Delta(G)-2$.*

Proof. Let $\Delta := \Delta(G)$. Since even-order graphs cannot be overfull, it suffices to show that, when n is odd, the following inequalities are equivalent:

$$|E(G)| > \Delta\left\lfloor\frac{n}{2}\right\rfloor; \tag{2.1}$$

$$\sum_{u\in V(G)}\Big(\Delta - d_G(u)\Big) \leqslant \Delta - 2\,. \tag{2.2}$$

But this equivalence follows immediately from the elementary fact that $\sum_{u\in V(G)} d_G(u) = 2|E(G)|$ for every graph G:

$$(2.2) \quad\Leftrightarrow\quad \Delta n - 2|E(G)| \leqslant \Delta - 2 \quad\Leftrightarrow\quad |E(G)| \geqslant \frac{\Delta(n-1)}{2}+1 \quad\Leftrightarrow\quad (2.1). \qquad \square$$

Notice that Observation 2.17 implies that if G is an overfull graph, then every major x of G is a *proper major*, that is, the local degree sum of x in G is at least $\Delta^2-\Delta+2$ (or, equivalently, $\sum_{ux\in E}(\Delta-d_G(u)) \leqslant \Delta-2$). Therefore, graphs with no proper majors are a special case of non-*SO* graphs. If the local degree sum of a major x in G is exactly $\Delta^2-\Delta+1$ (or, equivalently, if $\sum_{u\in N_G(x)}(\Delta-d_G(u)) = \Delta-1$), we say that x is a *tightly non-proper* major. If the local degree sum of x in G is strictly less than $\Delta^2-\Delta+1$ (or, equivalently, if $\sum_{u\in N_G(x)}(\Delta-d_G(u)) \geqslant \Delta$), we say that x is a *strictly non-proper* major. The graph class $\mathscr{X}$ introduced in Chapter 1, for which we present one of our main results (Theorem 1.10, p. 24), is the class of the graphs wherein all majors are strictly non-proper. Figure 2.9 extends Figure 1.16 (p. 25) showing the relations amongst the class $\mathscr{X}$, the non-*SO* graphs, and the *Class 1* graphs.

As observed by Beineke and Wilson (1973), being *SO* is clearly a sufficient condition for a graph to be *Class 2*. For some graph classes, such as graphs with $\Delta \geqslant n-3$ (Plantholt, 1981; Chetwynd and Hilton, 1984b,a, 1989b) and complete multipartite

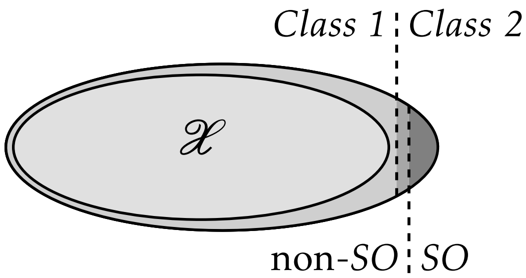


Figure 2.9: The graph class $\mathscr{X}$ and two partitions of the set of all graphs: according to Vizing's Theorem and according to the subgraph-overfullness property.

graphs (Hoffman and Rodger, 1992), this condition is also necessary[8]. In fact, the necessity is conjectured for all graphs with $\Delta > n/3$:

Conjecture 2.18 (Overfull Conjecture (Chetwynd and Hilton, 1984a, 1986; Hilton and Johnson, 1987)). *An n-vertex graph with maximum degree* $\Delta > n/3$ *is* Class 2 *if and only if it is subgraph-overfull.*

The Overfull Conjecture was first proposed by Chetwynd and Hilton (1984a), but stated for $\Delta \geqslant n/2$, in the work wherein they show that a graph with $\Delta \geqslant n-3$ and even n is *Class 2* if and only if it is *SO*. Two years later, the same authors restated the conjecture for $\Delta \geqslant n/3$ (Chetwynd and Hilton, 1986), but they soon found a non-*SO* *Class 2* graph with $\Delta = n/3$: the P^* (Figure 2.10), which is the graph obtained from the Petersen graph (Figure 1.7) by the removal of an arbitrary vertex. Then, in 1987, the

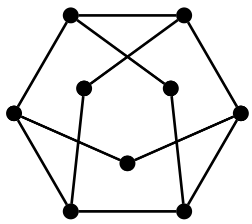


Figure 2.10: The graph P^*. Remark that, by the symmetry of the Petersen graph, all the graphs obtained from the Petersen graph by the removal of a vertex are isomorphic.

conjecture came to its current form, restricted to $\Delta > n/3$ (Hilton and Johnson, 1987).

The fact that both the Petersen graph (which has $\Delta = (n-1)/3$) and the P^* are *Class 2* non-*SO* graphs is a standard result which can be verified by observing that:

- they are *Class 2*, as shown in Observations 2.2 (p. 31) and 2.19 below;
- no snark can be *SO*, as shown in Observation 2.20.

Observation 2.19. *The graph P^* is not 3-edge-colourable.*

Proof. For the sake of contradiction, and in view of Theorem 2.8, take an equitable 3-edge-colouring of the P^*. Since the P^* has exactly 3 vertices of degree 2 and the

[8]In the case of graphs with $\Delta = n-1$ (i.e. graphs with a spanning star) and complete multipartite graphs, all *Class 2* graphs are actually overfull (Plantholt, 1981; Hoffman and Rodger, 1992).

other vertices have degree 3, each one of the 3 colours is missing at exactly one of the 3 vertices of degree 2. Ergo, if we add a vertex x to the P^* connecting it to the 3 vertices of degree 2, and if for each vertex u of degree 2 in the P^* we assign the colour missing at u to the edge ux, we obtain a 3-edge-colouring of the Petersen graph, contradicting Observation 2.2. □

Observation 2.20. *No snark can be* SO.

Proof. Since the sum of the degrees of a graph is always an even number, the order of a d-regular graph can never be odd if d is odd. Therefore, all snarks have even order and hence cannot be overfull. This implies that if a snark G is *SO*, then it has an overfull subgraph H with $\Delta(H) = 3$ and $V(H) \neq V(G)$, and we assume without loss of generality that H is induced by $V(H) =: U$. However, the connectedness of G along with Observation 2.17 bring that $|\partial_G(U)| = |\Delta(H)| - 2 = 1$, which contradicts the fact that snarks are bridgeless. □

Remark that the proof for Observation 2.19 can also be used to show that, for any snark G and any $u \in V(G)$, the graph $G - u$ is *Class 2* but not *SO*. However, since all snarks are cubic and the Petersen graph is the smallest snark, the P^* is the only such $G - u$ which maximises the ratio n/Δ, wherein $n := |V(G-u)|$. To the best of our knowledge, no *Class 2* non-*SO* graph with $\Delta > 3$ has been presented, even for constant Δ in function of n. In Chapter 4 we show that all *Class 2* d-regular complementary prism (like the Petersen graph, which is the 3-regular complementary prism $C_5\overline{C_5}$) are also non-*SO* graphs. Moreover, these graphs satisfies $d = (n+2)/4$, as the Petersen graph also does, and from each of these graphs the removal of any vertex would yield a *Class 2* non-*SO* graph with $\Delta = (n+3)/4$, like the P^*.

The equivalence between being *Class 2* and being *SO* is also conjectured for chordal graphs, even for n-vertex graphs with maximum degree $\Delta \leqslant n/3$ (Figueiredo et al., 2000). Actually, the authors conjecture a stronger statement: that all *Class 2* chordal graphs are *neighbourhood-overfull*. A graph G is said to be *neighbourhood-overfull* (shortly, *NO*) if there is some $u \in V(G)$ such that $\{u\} \cup N_G(u)$ induces an overfull Δ-subgraph of G. The conjecture by Figueiredo et al. (2000) has already been demonstrated for some subclasses of chordal graphs, such as split-indifference (Ortiz Z. et al., 1998) and, more recently, split-comparability (Sousa Cruz et al., 2017) graphs. Figueiredo et al. (2000) also showed that all *SO* indifference graphs are *NO*.

A k-factor of a graph G is a set $F \subseteq E(G)$ which induces a k-regular spanning subgraph. In particular, a 1-factor of G is a perfect matching in G. A k-factorisation of G is a partition of $E(G)$ into (disjoint) k-factors, possible only if G is kt-regular for some integer t. Clearly, a d-regular graph is *Class 1* if and only if it is 1-factorisable. Restricted to n-vertex d-regular graphs with $d \geqslant n/2$, the Overfull Conjecture is equivalent to the *1-Factorisation Conjecture*, which was stated by Chetwynd and Hilton (1985), but whose origin, according to the authors, may go back to G. A. Dirac in the early 1950s:

Conjecture 2.21 (1-Factorisation Conjecture (Chetwynd and Hilton, 1985)). *Every n-vertex d-regular graph with $d \geqslant n/2$ and n even is 1-factorisable.*

The equivalence between the restriction of the Overfull Conjecture to d-regular graphs with $d \geqslant n/2$ and the 1-Factorisation Conjecture comes from the fact that no n-vertex d-regular graph with $d \geqslant n/2$ and n even can be *SO* (Hilton, 1984; Niessen and Volkmann, 1990).

Remark that, when restricted to graph classes wherein subgraph-overfullness is a necessary condition for a graph to be *Class 2*, deciding chromatic index is a problem in $\mathcal{NP} \cap \text{co}\mathcal{NP}$: a polynomial-length certificate verifiable in polynomial-time for a *Class 2* graph in these graph classes can be simply an overfull Δ-subgraph. Moreover, the following results bring that under this restriction the problem is actually in $\mathcal{P}$, and justifies why there has been much work in the last 30 years aimed at identifying classes of graphs wherein all non-*SO* graphs are *Class 1*.

Theorem 2.22 (Padberg and Rao, 1982; Niessen, 1994, 2001). *The problem of deciding the existence of an overfull Δ-subgraph in a graph G with maximum degree Δ graph can be decided in polynomial time (Padberg and Rao, 1982). Moreover:*

(i) *if $\Delta \geqslant n/2$, then G has at most one induced overfull Δ-subgraph (Niessen, 1994);*

(ii) *if $\Delta > n/3$, then G has at most three induced overfull Δ-subgraphs (Niessen, 2001).*

We close this section highlighting that the upper bound in Observation 1.1 (p. 16) can be sharpened:

Observation 2.23. *Any graph G has at most $\chi'(G)\nu(G)$ edges.*

However, although having a Δ-subgraph H with more than $\Delta\nu(H)$ edges is in fact a sufficient condition for a graph to be *Class 2*, the following conjecture suggests that the bound $\chi'(G)\nu(G)$ in Observation 2.23 seems to be no sharper than $\chi'(G)\lfloor n/2 \rfloor$ for the definition of overfull and *SO* graphs in the context of the Overfull Conjecture:

Conjecture 2.24. *An n-vertex graph G with maximum degree $\Delta > n/3$ has a Δ-subgraph H satisfying $|E(H)| > \Delta\nu(H)$ if and only if G is* SO.

We could neither find a proof for Conjecture 2.24 in the literature, nor derive one by ourselves, but we remark that if Conjecture 2.24 does not hold, then neither does the Overfull Conjecture. Observation 2.25 is a straightforward evidence for Conjecture 2.24 in the case wherein $\Delta \geqslant (n-3)/2$.

Observation 2.25. *Every overfull n-vertex graph G with maximum degree $\Delta \geqslant (n-3)/2$ has a near-perfect matching, that is, satisfies $\nu(G) = \lfloor n/2 \rfloor$.*

Proof. From Vizing's Theorem (Theorem 1.4) follows that no graph with maximum degree Δ and matching number ν can have more than $(\Delta+1)\nu$ edges, since in a $(\Delta+1)$-edge-colouring each colour is assigned to at most ν edges. Hence, if G is an n-vertex overfull graph with maximum degree $\Delta \geqslant (n-3)/2$, then n is odd and we have

$$\nu(G) \geqslant \frac{|E(G)|}{\Delta+1} > \frac{\Delta}{\Delta+1}\left(\frac{n-1}{2}\right) \geqslant \frac{n-3}{n-1}\left(\frac{n-1}{2}\right) = \frac{n-3}{2},$$

which implies $\nu(G) \geqslant (n-1)/2 = \lfloor n/2 \rfloor$, since $\nu(G)$ is an integer. $\square$

3 A recolouring procedure for graphs with bounded local degree sums

In this chapter we present the following result on edge-colouring graphs with bounded local degree sums, as announced in Chapter 1, p. 24:

Theorem 1.10. *Let $\mathscr{X}$ be the class of the graphs with maximum degree Δ whose majors have local degree sum bounded above by $\Delta^2 - \Delta$. All graphs in $\mathscr{X}$ are* Class 1.

This chapter is organised as follows:

- Section 3.1 presents our extension of Vizing's recolouring procedure for Δ-edge-colouring graphs with bounded local degree sums;
- Section 3.2 presents the proof for Theorem 1.10, as well as further applications of our recolouring procedure;
- Section 3.3 closes the chapter with remarks on the time complexity of the algorithms presented in the proofs for the results in Sections 3.1 and 3.2, highlighting their polynomiality.

3.1 The recolouring procedure

Recall from Chapter 1 that Vizing's recolouring procedure requires every vertex of the recolouring fan to miss a colour. Although this is always possible in $(\Delta+1)$-edge-colourings, sometimes this is too restrictive when we are trying to use the procedure in the construction of a Δ-edge-colouring of a specific *Class 1* graph (recall Figure 1.15, p. 25). The proof for Theorem 2.3 is a successful example of a Δ-edge-colourability proof which considers the edges of the given graph in an appropriate order so that the condition for applying Vizing's recolouring procedure is always satisfied. Unfortunately, for other proofs this seems hard to manage. This section presents a recolouring procedure which extends Vizing's and does not require every vertex of the fan to miss a colour. Some simpler versions of this procedure we have published during during the development of this work. Since the current version completely covers the earlier ones, only the current version is presented in this text.

Throughout this section $G = (V, E)$ is a graph with n vertices and m edges and $\varphi\colon E \setminus \{uv\} \to \mathscr{C}$ is a Δ-edge-colouring of $G - uv$ for some $uv \in E$, which is the edge to be coloured by our recolouring procedure.

Definition 3.1 (*recolouring fan* and *virtually missing colours*). A sequence $v_0, \ldots, v_k$ of *distinct* neighbours of u in G is a *recolouring fan* for uv if $v_0 = v$ and, for all $i \in \{0, \ldots, k-1\}$:

- either v_i actually misses the colour $\alpha_i := \varphi(uv_{i+1})$;
- or v_i misses the colour $\alpha_i := \varphi(uv_{i+1})$ *virtually*, that is, $i > 0$ and $\varphi(v_i w_i) = \alpha_i$ for some $w_i \in N_G(v_i) \setminus \{v_{i-1}\}$ which actually misses α_{i-1}.

The recolouring fan is said to be *complete* if v_k misses, actually or virtually, a colour α_k missing at u.

Lemma 3.2. *If there is a complete recolouring fan for uv, then G is* Class 1.

Proof. We perform a procedure for i from k down to 0. At the beginning of each iteration it is invariant that both u and v_i miss α_i (the latter possibly virtually). So, we simply assign α_i to uv_i and, if v_i misses α_i virtually (which means that $i > 0$ and $\varphi(v_i w_i) = \alpha_i$ for some $w_i \in N_G(v_i) \setminus \{v_{i-1}\}$ which actually misses α_{i-1}), we also assign α_{i-1} to $v_i w_i$. If $i = 0$, we are done. If $i > 0$, now u misses α_{i-1}, which is still missing (possibly virtually) at v_{i-1}, so we can decrement i and continue. □

As for Vizing's recolouring procedure, the procedure described in the proof of Lemma 3.2 is referred to as the *decay of the colours* of the recolouring fan $v_0, \ldots, v_k$ (see Figure 3.1).

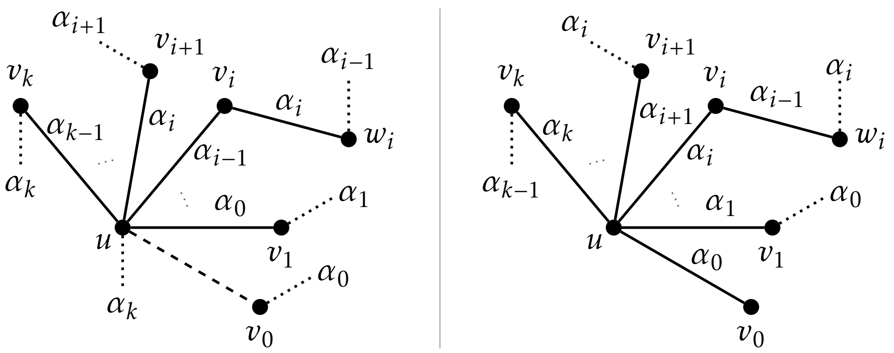


Figure 3.1: A complete recolouring fan before and after the decay of the colours

The vertices $v_0, \ldots, v_k$ of a recolouring fan are all distinct, which implies that $\alpha_0, \ldots, \alpha_{k-1}$ are all distinct colours not missing at u. Nevertheless, a vertex w_i may be equal to some vertex v_j (except to v_{i-1}, otherwise the decay of the colours would fail) or even to some other vertex w_j. The possibility of coincidences of the first kind emerges from the fact that we are not restricting ourselves to triangle-free graphs. It should also be noticed that, differently from the vertices $v_0, \ldots, v_k$, the vertex w_i is not necessarily defined for all $i \in \{0, \ldots, k\}$.

Lemma 3.3. *If there is a recolouring fan $v_0, \ldots, v_k$ for uv such that, for some $\beta \in \mathscr{C}$ missing at u and some $\alpha_k \in \mathscr{C} \setminus \{\alpha_0, \ldots, \alpha_{k-1}, \beta\}$ missing at v_k (the latter possibly virtually), either*

(i) *u and v_k do not lie in the same α_k/β-component, or*

(ii) *v_k misses α_k virtually and belongs to the same α_k/β-component, clearly a path, as u, and w_k is closer than v_k to u in this path,*

then G is Class 1.

Proof. If (i) holds, then exchanging the colours along the α_k/β-component X to which v_k belongs brings β missing at both u and v_k (the latter virtually if α_k was virtually missing at v_k). Since the colours $\alpha_0, \ldots, \alpha_k, \beta$ are all distinct, the colour exchanging operation does not compromise the recolouring fan and works even if X is a cycle (possible only if α_k was virtually missing at v_k). Now $v_0, \ldots, v_k$ is a complete recolouring fan for uv, so we apply Lemma 3.2 and we are done.

If (ii) holds, let P and P' be the α_k/β-paths starting at u in $G-uv$ and in $G-uv-v_kw_k$ respectively. Recall that P' is a subpath of P. Since w_k is closer than v_k to u in P, the outer vertices of P' are u and w_k, and v_k is not in P'. Hence, by temporarily uncolouring the edge v_kw_k, we are back to (i), so we exchange the colours along the α_k/β-path starting at v_k and we apply Lemma 3.2. After that, since $w_k(\neq v_{k-1})$ still misses α_{k-1} and the decay of the colours has transferred α_{k-1} from uv_k to uv_{k-1}, the uncoloured edge v_kw_k can be coloured α_{k-1}. □

Lemma 3.4. *Let $v_0,\dots,v_k$ be a recolouring fan for uv and*

$$G' := \begin{cases} G - v_{k-1}w_{k-1}, & \text{if } k>1 \text{ and } w_{k-1} \text{ is defined, or} \\ G, & \text{otherwise.} \end{cases}$$

Let also $\beta \in \mathscr{C}$ be a colour missing at u and $\alpha_k \in \mathscr{C}$ be a colour virtually missing at v_k. If:

(i) $w_k \in N_G(u)$, *and*

(ii) $\gamma := \varphi(uw_k) \neq \alpha_i$ *for all* $i \in \{0,\dots,k-1\}$, *and*

(iii) *w_k is not in the same $\alpha_{k-1}/\varphi(uw_k)$-component of G' as v_{k-1} after exchanging the colours along the α_{k-1}/β-component of G' to which w_k belongs,*

then G is Class 1.

Proof. First of all, we check if the recolouring fan $v_0,\dots,v_{k-1}$ for uv already satisfies Lemma 3.3, because, if it does, we are done. If it does not, we uncolour the edge $v_{k-1}w_{k-1}$, in the case wherein w_{k-1} is defined. Then, we exchange the colours along the α_{k-1}/β-component P of G' to which w_k belongs (clearly a path). Since neither u nor $v_{k-1}(\neq w_k)$ is in P, otherwise $v_0,\dots,v_{k-1}$ would satisfy Lemma 3.3, we get β missing at both u and w_k (see Figure 3.2(a)).

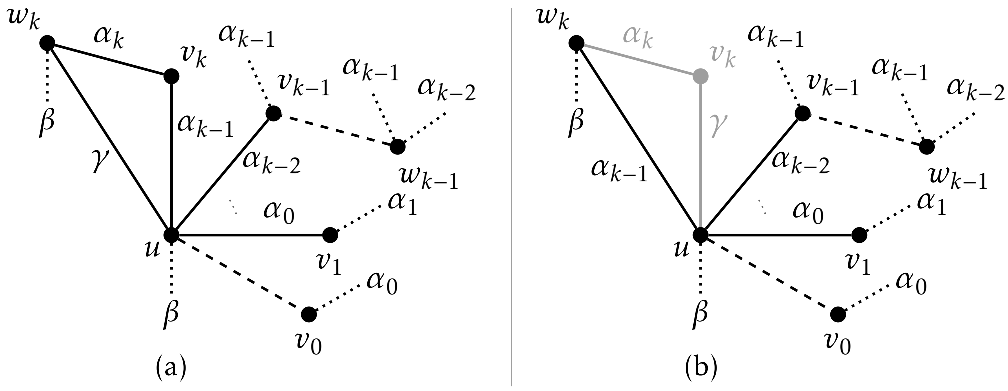


Figure 3.2: An illustration for the proof of Lemma 3.4. Here and in the next figures, differently from how we have used waved curves in the figures which illustrate Vizing's recolouring procedure in Chapter 1, we do not depict the underlying colour-alternating paths any more, so the illustration can be presented as clean as possible.

Let X be the α_{k-1}/γ-component to which u, v_k, and w_k belong (not necessarily a path, as we have exchanged the colours along P). Since, by hypothesis, v_{k-1} is not in X and $\gamma \neq \alpha_i$ for all $i \in \{0,\dots,k-1\}$, exchanging the colours along X makes $v_0,\dots,v_{k-1},w_k$ a complete recolouring fan (see Figure 3.2(b)). Then, we apply Lemma 3.2 and assign to the edge $v_{k-1}w_{k-1}$, if any, the colour α_{k-2}. □

Let $v_0,\ldots,v_k$ be a maximal, but not complete, recolouring fan for uv. Extending Lemma 1.8 (p. 20), we show that if v_k misses, actually or virtually, the colour α_j for some $j<k$ (which implies $j<k-1$, since $\alpha_{k-1}=\varphi(uv_k)$), then G is *Class 1*. We split the proof into three lemmas, each handling one of the following cases:

1. either v_{j+1} actually misses α_{j+1} or there is some $\beta\in\mathscr{C}$ missing at u such that w_{j+1} is not in the same α_j/β-component as v_k (Lemma 3.5);
2. the vertices v_{j+1} and v_k are neighbours, the vertex v_{j+1} misses α_{j+1} virtually, and v_k is the neighbour of v_{j+1} which actually misses α_j selected for the role of w_{j+1} (Lemma 3.6);
3. the vertex v_{j+1} misses α_{j+1} virtually and w_{j+1} is in the same α_j/β-component W as v_k for all $\beta\in\mathscr{C}$ missing at u (Lemma 3.7).

Remark that the second case (see Figure 3.3) is a particular case of the third.

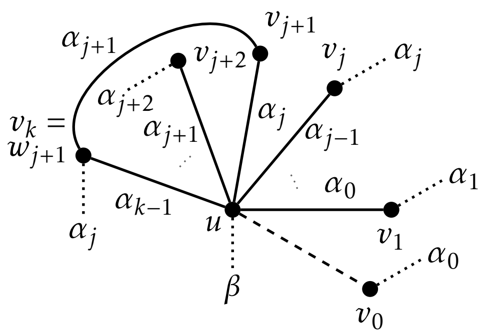


Figure 3.3: A recolouring fan in which v_k misses α_j for some $j<k$, the vertex v_{j+1} misses α_{j+1} virtually, and $v_k=w_{j+1}$ (possible only if v_k actually misses α_j). Here the colours α_j and α_{j+2} are actually missing at v_j and v_{j+2}, respectively, but they could be missing virtually.

Lemma 3.5. *If there is a recolouring fan $v_0,\ldots,v_k$ for uv such that v_k misses, actually or virtually, the colour α_j for some $j<k$, and if:*

- *either v_{j+1} actually misses α_{j+1},*
- *or there is some $\beta\in\mathscr{C}$ missing at u such that w_{j+1} is not in the same α_j/β-component as v_k,*

then G is Class 1.

Proof. Recall that $j<k-1$. We first check if the recolouring fan $v_0,\ldots,v_j$ satisfies Lemma 3.3, in which case we are already done. If it does not, we uncolour v_kw_k, in the case wherein w_k is defined, leaving this edge to be coloured later. Then, let $\beta\in\mathscr{C}$ be a colour missing at u and let P be the α_j/β-path starting at v_k. If w_k is defined, let Q be the α_j/β-path starting at w_k.

We know that u and v_j are in the same α_j/β-path, otherwise $v_0,\ldots,v_j$ would satisfy Lemma 3.3. Therefore, either both u and v_j are in P (possible only if v_j misses α_j virtually and the outer vertices of P are u and v_k), or none amongst u and v_j is in P. We have the following cases:

1. If neither u nor v_j is in P, then exchanging the colours of the edges along P yields β as a colour missing at both u and v_k. Hence, we get $v_0,\dots,v_k$ as a complete recolouring fan for uv, even if v_{j+1} misses α_{j+1} virtually, in which case w_{j+1} is not in P by hypothesis and still misses α_j as desired. So, we apply Lemma 3.2 to colour uv. Then, if w_k is defined, the edge v_kw_k can be coloured α_{k-1} and we are done.

2. If both u and v_j are in P, then v_j is closer than w_j to u in P, otherwise $v_0,\dots,v_j$ would satisfy Lemma 3.3. Hence, by temporarily uncolouring the edge v_jw_j, the α_j/β-path containing v_k becomes a subpath P' of P with v_k and w_j as outer vertices. Exchanging the colours along P', we get β missing at both u and v_k, thus $v_0,\dots,v_k$ becomes a complete recolouring fan for uv. We apply Lemma 3.2 on this fan and then the temporarily uncoloured edges v_jw_j and (if w_k is defined) v_kw_k can be coloured α_{j-1} and α_{k-1}, respectively. □

Remark that the recolouring fan constructed for the edge uv_0 in the crab claw graph in Figure 1.15 (p. 25), which we could not manage with Vizing's recolouring procedure, can be managed with our recolouring procedure. Furthermore, it satisfies the conditions of Lemma 3.5: the vertex v_3 has a neighbour w_3 which misses the colour $\alpha_2 = \varphi(uv_3) = 4$, that is, v_3 *virtually* misses the colour $\varphi(v_3w_3) = 2 = \alpha_1$, and this colour is actually missing at v_1 (see Figure 3.4(a)). Observing that the only colour β missing at

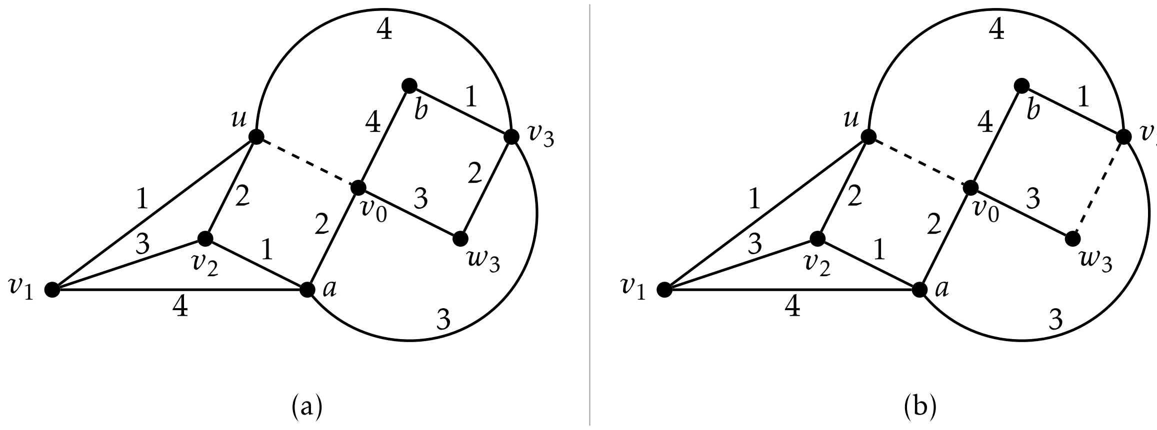


Figure 3.4: An illustration for the first steps of the proof of Lemma 3.5

u is 3 and that u and v_1 are in the same α_1/β-path, we perform the proof of the lemma for this recolouring fan:

1. We uncolour the edge v_3w_3 (see Figure 3.4(b)).

2. Being $P = v_3av_0w_3$ the α_1/β-path starting at v_3, we fall into the case wherein none amongst u and v_1 is in P. Then, following the proof, we exchange the colours along P (see Figure 3.5(a) on the next page). Remark that after this colour exchanging operation the colour α_2, not involved in the operation, is still missing at the vertex w_3, as we need.

3. We apply the decay of the colours to colour uv_0 and then we assign to the edge v_3w_3 the colour α_2 (see Figure 3.5(b)).

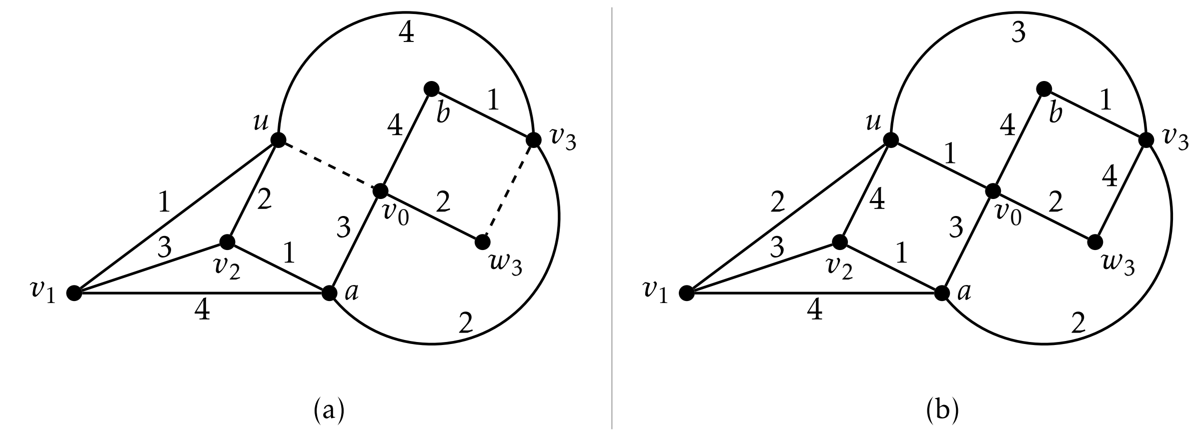


Figure 3.5: An illustration for the last steps of the proof of Lemma 3.5

Lemma 3.6. *If there is a recolouring fan $v_0, \ldots, v_k$ for uv and some $j < k$ such that v_{j+1} misses α_{j+1} virtually and $w_{j+1} = v_k$, then G is* Class 1.

Proof. First, we verify:

1. if the recolouring fan $v_0, \ldots, v_j$ for uv satisfies Lemma 3.3;
2. if the recolouring fan $v_0, \ldots, v_{j+1}$ for uv satisfies Lemma 3.4, observing that $\varphi(uw_{j+1})(= \alpha_{k-1}) \neq \alpha_i$ for all $i \in \{0, \ldots, j\}$ already holds.

If the verification above have failed, we uncolour the edge $v_j w_j$, in the case wherein v_j misses α_j virtually, leaving this edge to be coloured later (see Figure 3.6(a)). Then, let β be a colour of $\mathscr{C}$ missing at u and let P be the α_j/β-path starting at v_k. Since we have assumed that Lemma 3.3 is not satisfied by $v_0, \ldots, v_j$, and since the edge $v_j w_j$ has been uncoloured in the case wherein v_j misses α_j virtually, we know that neither u nor v_j is in P. We perform the following steps:

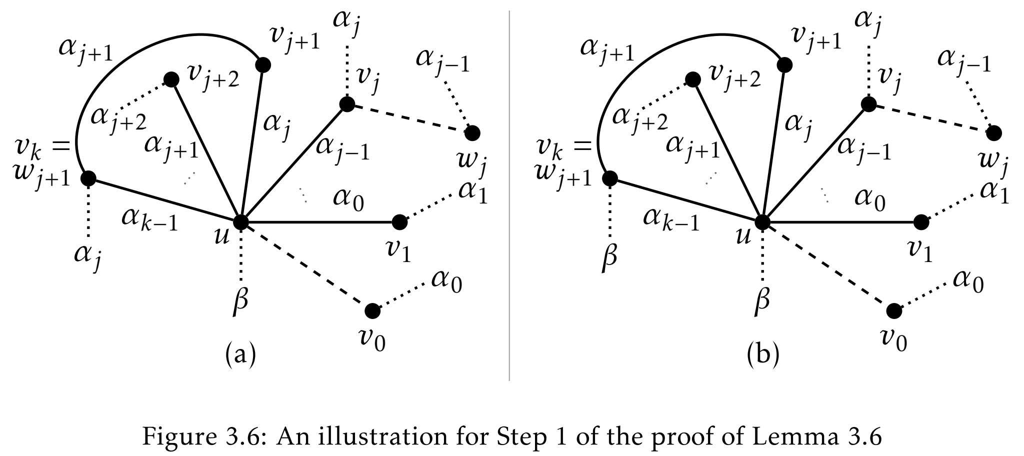


Figure 3.6: An illustration for Step 1 of the proof of Lemma 3.6

Step 1. We exchange the colours along P. The colour exchanging operation yields β missing at both u and v_k, but now $v_0, \ldots, v_k$ is no longer a recolouring fan, since $w_{j+1}(= v_k)$ no longer misses α_j (see Figure 3.6(b)).

Step 2. Recall that u and v_j are in the same α_j/α_{k-1}-component, otherwise Lemma 3.4 would have been satisfied by $v_0, \ldots, v_{j+1}$ at the beginning of the proof. We perform the decay of the colours described in the proof of Lemma 3.2, starting at the edge uv_k, but stopping just before the procedure reaches uv_{j+1}. Then, we temporarily uncolour $v_{j+1}w_{j+1}$ and assign α_{j+1} to uv_{j+1}. Now, the colour α_j is missing at both u and v_j (see Figure 3.7(a)), but we cannot apply Lemma 3.2 on $v_0,\dots,v_j$ (yet), because this would leave $v_{j+1}v_k$ uncoloured.

Step 3. Let Q be the α_j/α_{k-1}-path starting at v_k. Remark that Q cannot end at v_{j+1}, otherwise in the beginning of Step 2 the α_j/α_{k-1}-component containing u would have been a cycle, thus not containing v_j, a contradiction. Therefore, now, by exchanging the colours along Q we obtain α_j missing at v_k again and still at v_{j+1}, so $v_{j+1}v_k$ can be coloured α_j. If neither u or v_j is in Q, then α_j is still missing at both u and v_j, so we can apply Lemma 3.2 on $v_0,\dots,v_j$ (assigning to the edge v_jw_j, if any, the colour α_{j-1}) and we are done. However, it might be the case that one amongst u and v_j is the other outer vertex of Q, which would make α_j not be missing at either u (as in Figure 3.7(b)) or v_j.

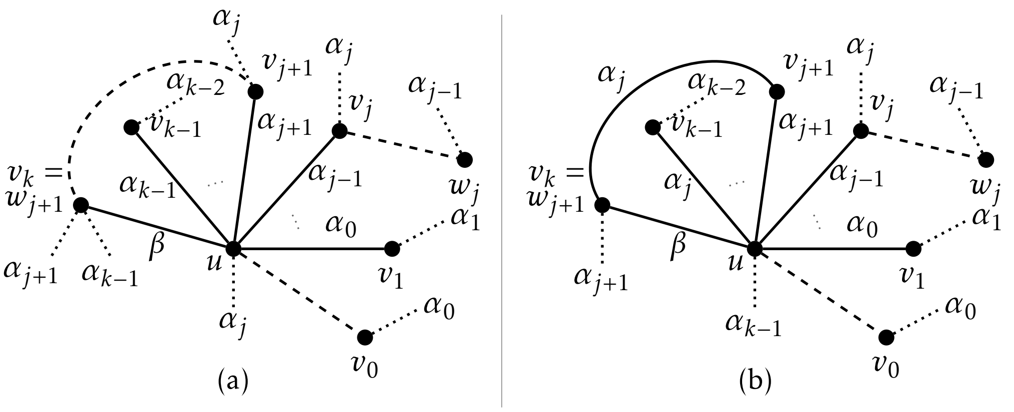


Figure 3.7: An illustration for Steps 2 and 3 of the proof of Lemma 3.6

Step 4. For this step we have the following cases, in each of which we end up with a recolouring fan satisfying the conditions of either Lemma 3.2 or Lemma 3.5. Notice in all the cases that, after applying the corresponding lemma, the edge v_jw_j, if any, can be coloured α_{j-1}.

(a) If the vertices u and v_j still do not lie in the same α_j/α_{k-1}-component, even after assigning α_j to $v_{j+1}v_k$ in Step 3, then changing the colours along the component to which one of them belongs yields the same colour missing at both. This way $v_0,\dots,v_j$ becomes a complete recolouring fan and we are done by Lemma 3.2.

(b) If assigning α_j to $v_{j+1}v_k$ in Step 3 connected the α_j/α_{k-1}-component to which u and v_j belonged, then the sequence

$$v_0, v_1, \dots, v_j, v_{k-1}, v_{k-2}, \dots, v_{j+2}, v_{j+1}$$

becomes a recolouring fan whose last vertex (v_{j+1}) virtually misses a colour (α_j) which is missing at either u (see Figure 3.8(a) on the next page), satisfying Lemma 3.2, or at another vertex (v_j) in the fan (see Figure 3.8(b)), satisfying Lemma 3.5. The former or the latter happens if, in Step 3, the other outer vertex of Q beside v_k was v_j or u, respectively. □

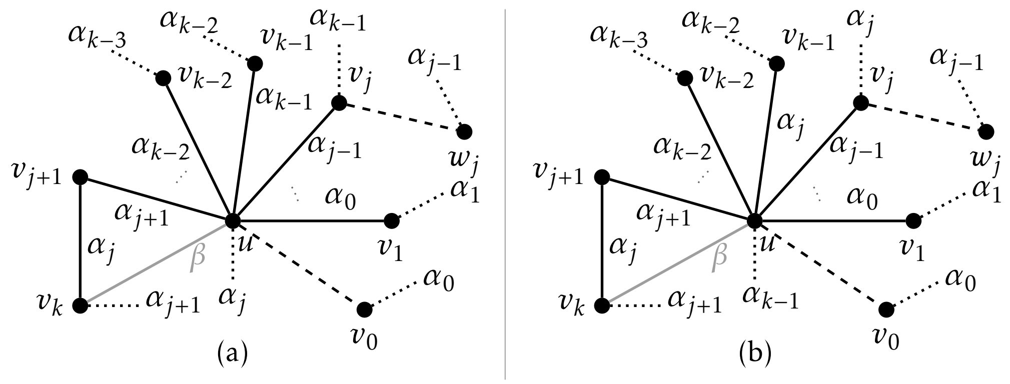


Figure 3.8: An illustration for Step 4 of the proof of Lemma 3.6

Lemma 3.7. *If there is a recolouring fan $v_0,\dots,v_k$ for uv and some $j<k$ such that v_{j+1} misses α_{j+1} virtually and w_{j+1} is in the same α_j/β-component W as v_k for all $\beta\in\mathscr{C}$ missing at u, then G is* Class 1.

Proof. Let $\beta\in\mathscr{C}$ be a colour missing at u. In view of Lemma 3.6, we assume $w_{j+1}\neq v_k$, and also that the recolouring fan $v_0,\dots,v_j$ for uv does not satisfy Lemma 3.3, since otherwise we would already be done. Recall that W is clearly a path.

The proof goes similarly to the proof of Lemma 3.6. We first uncolour v_jw_j, if w_j is defined. Then, let P be the α_j/β-component to which v_k belongs. Observe that $P=W$ or P is a subpath of W. Since we have assumed that Lemma 3.3 is not satisfied by $v_0,\dots,v_j$, and since the edge v_jw_j, if any, has been uncoloured, we know that neither u nor v_j is in P.

It should be noticed that the uncolouring of v_jw_j, if w_j is defined, may have led to v_k and w_{j+1} not belonging to the same α_j/β-path any more. If that is the case, we proceed exactly as in the proof of Lemma 3.5 and we are done. Otherwise, we perform the following steps:

Step 1. We uncolour the edge v_kw_k, if any, and we exchange the colours along the α_j/β-path starting at v_k (a subpath Q of P). This yields β missing at u and also at v_k. If w_{j+1} is not in Q (which is possible due to the uncolouring of v_kw_k), then $v_0,\dots,v_k$ is a complete recolouring fan, so we apply Lemma 3.2, assigning α_{k-1} to v_kw_k, and we are done. Otherwise, the sequence $v_0,\dots,v_k$ is no longer a recolouring fan, since w_{j+1} is in P and no longer misses α_j.

Step 2. We perform the decay of the colours starting at uv_k, but stopping just before uv_{j+1}, transferring the colour α_{j+1} from $v_{j+1}w_{j+1}$ to uv_{j+1}, assigning α_{k-1} to v_kw_k (if any), and leaving $v_{j+1}w_{j+1}$ temporarily uncoloured, as in Step 2 of the proof of Lemma 3.6. Now, α_j is missing at u, v_j, and v_{j+1}, and β is missing at w_{j+1} and possibly virtually at v_k.

Step 3. Let P' be the α_j/β-path which starts at w_{j+1} now. If β is virtually missing at v_k and if w_k was closer to w_{j+1} in P than v_k, then the outer vertices of P' are w_{j+1} and w_k. Otherwise, the outer vertices of P' are w_{j+1} and u. In neither case can v_j or v_{j+1} be in P', since they both miss α_j and the outer vertices of P' both miss β. So, we exchange the colours along P' and assign α_j to $v_{j+1}w_{j+1}$. If u is not in P', then it still misses α_j and $v_0,\dots,v_j$ fits the conditions of Lemma 3.2. If u is in P, then it now misses β, whilst v_j still misses α_j, and the sequence

$$v_0, v_1, \ldots, v_j, v_k, v_{k-1}, \ldots, v_{j+2}, v_{j+1}$$

becomes a recolouring fan which fits the conditions of Lemma 3.5, with α_j virtually missing v_{j+1}. After applying the corresponding lemma, the edge $v_j w_j$, if any, can be coloured α_{j-1}. □

Lemma 3.8 below concludes the presentation of our recolouring procedure.

Lemma 3.8. *If all majors of $G - uv$ adjacent to u in G are strictly non-proper majors of G, then G is* Class 1.

Proof. Throughout this proof, let G' denote the subgraph of G induced by the edges coloured by the moment. Since the vertices u and v are not majors of $G - uv$, each of them actually misses a colour of $\mathscr{C}$, meaning that $v_0 = v$ is by itself a recolouring fan for uv. If this recolouring fan already satisfies some amongst Lemmas 3.2–3.7, we are done. Otherwise, we continue its construction, until some of the aforementioned lemmas is satisfied, or until we have reached a maximal recolouring fan $v_0, \ldots, v_k$. We assume that, in this recolouring fan:

- only majors of $G - uv$ are allowed to virtually miss a colour;
- for all $j < k-1$, v_j does not miss α_{k-1}, actually or virtually, otherwise we would be able to replace the current recolouring fan with $v_0, \ldots, v_j, v_k$.

The only possible reason why the recolouring fan $v_0, \ldots, v_k$ is maximal and does not satisfy any amongst Lemmas 3.2–3.7 is if v_k is a major of $G - uv$ (thus no colour is actually missing at v_k) and one of the following cases holds:

Case 1. there is a colour β actually missing at both u and at some $y \in N_{G'}(v_k)$;

Case 2. no neighbour of v_k in $G' - v_{k-1}$ actually misses α_{k-1} and no neighbour of v_k in $G' - u$ misses β, for all $\beta \in \mathscr{C}$ missing at u.

Subroutine for Case 1. In this case, we first check if y is the vertex v_{k-1}. If it is, then $v_0, \ldots, v_{k-1}$ satisfies Lemma 3.2 and we are done. If $y \neq v_{k-1}$, we know that no neighbour of v_k in G', except for v_{k-1} if $v_{k-1}v_k \in E$, misses α_{k-1}, otherwise either $v_0, \ldots, v_k$ would satisfy some amongst Lemmas 3.2–3.7 or would not be maximal. Let P be the α_{k-1}/β-path starting at u. If v_{k-1} is not in P, then $v_0, \ldots, v_{k-1}$ satisfies Lemma 3.3 and we are done. Otherwise, we have the following subcases:

1. If y is not in P, we exchange the colours along P. Observe that $v_0, \ldots, v_k$ remains a recolouring fan, because the colour of the edge uv_k (β) is missing (possibly virtually) at v_{k-1} and, for all $j < k-1$, neither α_{k-1} nor β is equal to α_j. However, now v_k has a neighbour (y) distinct from v_{k-1} which misses the colour of the edge uv_k (β), which means that v_k virtually misses the colour of the edge $v_k y$, with y in the role of w_k. Remark that if we have not guaranteed that $y \neq v_{k-1}$ then y would not be able to assume the role of w_k (recall Definition 3.1). Therefore, being α_k be the colour of $v_k y$, either the recolouring fan $v_0, \ldots, v_k$ satisfies the conditions of some amongst Lemmas 3.2–3.7, or there is some $v_{k+1} \in N_{G'}(u)$ such α_k is the colour of the edge uv_{k+1}. In the former case, we are done. In the latter, the recolouring fan is no longer maximal and we can continue to construct it.

2. If y is in P, then v_{k-1} (which is also in P) misses α_{k-1} virtually and the outer vertices of P are u and y. If w_{k-1} is closer than v_{k-1} to u in P, then $v_0,\dots,v_{k-1}$ satisfies Lemma 3.3 and we are done. Otherwise:

   (a) If $\varphi(v_k y) = \alpha_j$ for some $j < k$ (which implies $j < k-1$), we check if $v_0,\dots,v_j$ satisfies Lemma 3.3. We also check if exchanging the colours along the α_j/β-component containing y makes $v_0,\dots,v_{k-1}$ satisfy Lemma 3.3. If it does not, we undo this colour exchanging operation and uncolour the edges $v_k y$ and $v_{k-1}w_{k-1}$. Now v_k actually misses α_j and $v_0,\dots,v_k$ satisfies Lemma 3.5. Since $v_0,\dots,v_j$ does not satisfy Lemma 3.3, performing the proof of Lemma 3.5 leads to the decay of the colours of the recolouring fan $v_0,\dots,v_k$, after which we have $\varphi(uv_k) = \beta$ and $\varphi(uv_{k-1}) = \alpha_{k-1}$, so $v_{k-1}w_{k-1}$ can be coloured α_{k-2}. Then, exchanging the colours along the α_{k-1}/β-component Q which contains y (which may have been modified while we performed the proof of Lemma 3.5) brings α_{k-1} missing at y. Since v_k is not in Q (otherwise in the beginning the colour exchanging operation on the α_j/β-component to which y belonged would have made $v_0,\dots,v_{k-1}$ satisfy Lemma 3.3), the edge $v_k y$ can be coloured α_{k-1} and we are done.

   (b) If $\varphi(v_k y) \notin \{\alpha_0,\dots,\alpha_{k-1}\}$, we *pretend* that v_k misses $\alpha_k := \varphi(v_k y)$ virtually, with y in the role of w_k, and we continue to construct the recolouring fan, until we get a recolouring fan which satisfies some amongst Lemmas 3.2–3.7. Performing the proof of the corresponding lemma leads to a decay of colours which may not *pass by* the edge uv_k. If that happens, we have nothing to handle. However, if the decay of colours which is to be performed shall *pass by* uv_k, we first check if it is still true that $v_0,\dots,v_{k-1}$ does not satisfy Lemma 3.3. Then, when the decay of the colours reaches the edge uv_k, we uncolour $v_{k-1}w_{k-1}$ and exchange the colours along the α_{k-1}/β-path starting at y (which does not end at v_{k-1}). After that, we proceed normally with the decay, obtaining $\varphi(uv_k) = \alpha_k$, $\varphi(v_k y) = \alpha_{k-1}$, $\varphi(uv_{k-1}) = \alpha_{k-1}$, and $\varphi(v_{k-1}w_{k-1}) = \alpha_{k-2}$.

Subroutine for Case 2. In this case, first observe that, since v_k is a strictly non-proper major of G and uv has not been coloured yet, we have

$$\sum_{y\in N_{G'}(v_k)} (\Delta - d_{G'}(y)) \geqslant \Delta + 1\,. \tag{3.1}$$

Hence, by the Pigeonhole Principle, there must be some $\gamma \in \mathscr{C}$ actually missing at two neighbours y_1 and y_2 of v_k in G'. This colour γ cannot be α_{k-1} nor β, by the hypothesis of the case. Further, only one amongst y_1 and y_2, say y_1, is possible to be in the β/γ-path which ends at u, so we exchange the colours along the β/γ-path starting at y_2 belongs. We know that γ is neither α_{k-1} nor missing at u, and that v_{k-1} and u lie in the same α_{k-1}/β-path. However, it is possible that $y_2 = v_j$ for some $j \in \{0,\dots,k-1\}$. If that is the case, we have now β missing at both u and v_j, so $v_0,\dots,v_j$ is a complete recolouring fan and we are done by Lemma 3.2. Otherwise, we are back to Case 1. □

3.2 Main result and further applications

Now we are ready to prove our main result on edge-colouring graphs with bounded local degree sums.

Proof of Theorem 1.10 (p. 24). With the recolouring procedure which we have developed in Section 3.1, the proof follows straightforwardly by constructing a Δ-edge-colouring $\varphi\colon E(G) \to \mathscr{C}$ edge by edge, applying Lemma 3.8 at each edge uv considered, since in all graphs in $\mathscr{X}$ all majors have local degree sum bounded above by $\Delta^2 - \Delta$, that is, are strictly non-proper. □

In Theorem 3.9 below we show that almost every graph is in $\mathscr{X}$ even given that the graph has cycles in the core, as we have announced in Chapter 1. In the proof, as it is usual in the context of random graphs[1], when we say that some set $S \subseteq V(G) \cup E(G)$ in an n-vertex $\mathscr{G}(n,1/2)$ graph G *almost surely* satisfies some property $p(S)$, this means that $\mathbb{P}[p(S)] \to 1$ as $n \to \infty$.

Theorem 3.9. *Almost every graph which has cycles in the core and $\Delta > 2$ is in $\mathscr{X}$.*

Proof. Let G be a $\mathscr{G}(n,1/2)$ graph. Consider the following:

Claim. There is some constant k_0 such that:

(i) G almost surely has a vertex of degree at least k_0;

(ii) for every $u \in V(G)$ such that $d_G(u) > k_0$, the graph G almost surely does not have a vertex of degree exactly $d_G(u) - 1$.

If the claim holds and if u is a uniformly sampled vertex from $V(G)$, then $\mathbb{P}[d_G(u) \leqslant \Delta - 2] \to 1$ as $n \to \infty$ if it is given that $\Delta > 2$. Furthermore, by linearity of expectation follows that if it is also given that the vertices of some $U \subseteq V(G)$ have degree Δ, then still the vertices of $V(G) \setminus U$ almost surely have degree at most $\Delta - 2$. Hence, if it is given that $\Lambda[G]$ has a cycle C and $\Delta > 2$, then almost surely $\Lambda[G] = C$ and hence, for each $x \in \Lambda[G]$, all neighbours of x which are not in C almost surely have degree in G at most $\Delta - 2$. This brings that almost surely

$$\sum_{u \in N_G(x)} d_G(u) \leqslant 2\Delta + (\Delta - 2)^2 \leqslant \Delta^2 - \Delta,$$

as we wanted.

Now we shall prove that the claim holds, following the same argument by Erdős and Wilson (1977) used to show that there is some constant k_0 such that G almost surely has a vertex of degree at least k_0, but, for every $k > k_0$, the graph G almost surely does not have two vertices both of degree k. In fact, this was shown by Erdős and Wilson for

$$k_0 := \frac{1}{2}\Big((n-1) + (1-\varepsilon)n\ln n\Big)^{\frac{1}{2}} \tag{3.2}$$

for any constant $\varepsilon > 0$. We shall demonstrate that Erdős and Wilson's proof also works to show that, for every $k > k_0$, the graph G almost surely does not have two vertices u and v such that $d_G(u) = k$ and $d_G(u) = k - 1$.

[1] Again, we refer the reader to Bollobás (2001) for terms and concepts on random graphs.

Given two vertices u and v and two integers k_1 and k_2, a classical elementary fact on random graphs is that the probability that the events $d_G(u) = k_1$ and $d_G(v) = k_2$ both occur, despite their non-independence, is (by the Inclusion–Exclusion Principle)

$$(1+o(1))\binom{n-1}{k_1}\binom{n-1}{k_2}2^{2-2n}.$$

Therefore, being any $\varepsilon > 0$ and k_0 defined for ε as in (3.2), the probability of the event E that a *labelled* $\mathscr{G}(n,1/2)$ graph G has two vertices u and v such that $d_G(u) = k > k_0$ and $d_G(v) = k-1$ is

$$\mathbb{P}(E) = (1+o(1))n^2 \sum_{k>k_0} \binom{n-1}{k}\binom{n-1}{k-1}2^{2-2n} = (1+o(1))n^2 \sum_{k>k_0} \frac{k}{n-k}\binom{n-1}{k}^2 2^{2-2n}.$$

As shown by Erdős and Wilson (1977),

$$\binom{n-1}{k}2^{1-n} \to 0 \quad \text{as } n \to \infty \quad \text{if } k > \frac{1}{2}\Big((n-1)+(1-\varepsilon)n\ln n\Big)^{\frac{1}{2}}.$$

Hence, $\mathbb{P}(E) \to 0$ as $n \to \infty$.

It remains to demonstrate, being E' the event that a *unlabelled* $\mathscr{G}(n,1/2)$ graph G has two vertices u and v such that $d_G(u) = k > k_0$ and $d_G(u) = k-1$, that also $\mathbb{P}(E') \to 0$ as $n \to \infty$. But this follows from a classical result in random graph theory according to which every property which holds for almost every labelled graph simultaneously holds for almost every unlabelled graph (Harary and Palmer, 1973, Chapter 9). □

In earlier versions of the recolouring procedure presented in Section 3.1, we have restricted ourselves to triangle-free graphs first, until we could manage the many complications originated by allowing triangles and the procedure could thus come to its current version. Of course, all the lemmas in Section 3.1 still hold for the restricted case of triangle-free graphs. Nevertheless, Lemma 3.8 in particular can be stated a slightly stronger for triangle-free graphs:

Lemma 3.10. *Let $G = (V,E)$ be a triangle-free graph and $\varphi\colon E\setminus\{uv\} \to \mathscr{C}$ be a Δ-edge-colouring of $G-uv$ for some $uv \in E$. If all majors of $G-uv$ adjacent to u in G are (not necessarily strictly) non-proper majors of G, then G is* Class 1.

Proof. The same proof for Lemma 3.8 applies, with a minor modification in Case 2, when we have reached a maximal recolouring fan $v_0,\dots,v_k$ such that no neighbour of v_k in $G'-v_{k-1}$ actually misses α_{k-1} and no neighbour of v_k in $G'-u$ misses β, for all $\beta \in \mathscr{C}$ missing at u. Remark that, in the case of triangle-free graphs, saying that no neighbour of v_k in $G'-v_{k-1}$ actually misses α_{k-1} is the same as saying that no neighbour of v_k in G' actually misses α_{k-1}. From this remark, from $\sum_{y\in N_{G'}(v_k)}(\Delta - d_{G'}(y)) \geqslant \Delta$, and from the Pigeonhole Principle, there must be some $\gamma \in \mathscr{C}\setminus\{\alpha_{k-1},\beta\}$ actually missing at two neighbours y_1 and y_2 of v_k in G', and the rest of the proof follows analogously as the proof for Lemma 3.8. □

From Lemma 3.10 follows a stronger version of Theorem 1.10 when restricted to triangle-free graphs:

Theorem 3.11. *Every triangle-free graph with no proper majors is* Class 1.

Proof. Analogous to proof for Theorem 1.10, but applying Lemma 3.10 instead of Lemma 3.8 at each edge uv considered. □

The first application of the earliest version of our recolouring procedure was on edge-colouring triangle-free graphs with $\Delta \geqslant n/2$ (see Theorem 3.12), which we show to be *Class 1*. Although one can derive an alternative and much simpler proof for this fact, without requiring an extended recolouring procedure, we consider that the main contribution in this result is the novel recolouring procedure itself, regardless of the simple application presented.

Theorem 3.12. *Every triangle-free graph with $\Delta \geqslant n/2$ is* Class 1.

Proof. We shall construct an edge-colouring of G edge by edge using a set $\mathscr{C}$ with Δ colours. For each edge uv taken in order to receive a colour, if we define G' as the subgraph of G induced by the edges coloured by the moment, we claim that, for every neighbour x of u in G' which has degree Δ in G' (and thus miss actually no colour), either no recolouring fan for uv reaches x, or there is some $w \in N_{G'}(x)$ which actually misses the colour α of the edge ux. If this claim holds, it is clear that we can always construct a recolouring fan for uv which satisfies Lemma 3.2 or Lemma 3.5.

Now we shall prove that α is actually missing at some $w \in N_{G'}(x)$ in the case wherein x is reachable by a recolouring fan for uv, as claimed. Let F (clearly a matching) be the set of all α-coloured edges and k be the number of neighbours of x which do not miss α. We have the following cases:

1. If there is some $y \in N_{G'}(u)$ which actually misses α, then $n-1 \geqslant 2|F|$. Further, since G is triangle-free, no edge of F can connect two neighbours of x, and thus $|F| \geqslant k$. However, if we assume that every neighbour of x is an end-vertex of an α-coloured edge, we have $k = \Delta \geqslant n/2$ and, by consequence, $n-1 \geqslant 2(n/2)$, a contradiction.

2. If no neighbour of u in G' actually misses α and, again for the sake of contradiction, to every neighbour of x incides an α-coloured edge, then $\Delta = n/2$ and F is a perfect matching on G, since G is triangle-free. Because there is a recolouring fan $v_0,\ldots,v_k$ for uv such that $v_k = x$, then $\alpha = \alpha_{k-1}$ and v_{k-1} misses α virtually, implying that some neighbour w_{k-1} of v_{k-1} in G' actually misses α_{k-2}. Therefore, the component of $G'[\alpha,\alpha_{k-2}]$ to which w_{k-1} belongs is a path P. If the number of edges in P is even, then the other outer vertex of P besides w_{k-1} is a vertex at which α is actually missing, contradicting the fact that F is a perfect matching on G. On the other hand, if the number of edges in P is odd, then exchanging the colours along P yields an edge-colouring of G' in which the colour α is assigned to $n/2+1$ edges, something impossible to happen. □

As we have mentioned, there is an alternative and much simpler proof for Theorem 3.12, without requiring an extended recolouring procedure. It follows immediately from the observation below and from the Δ-edge-colourability of bipartite graphs (Theorem 1.3, p. 18) and graphs with acyclic core (Theorem 2.3, p. 31).

Observation 3.13. *Every triangle-free graph with $\Delta \geqslant n/2$ is bipartite or its core is edgeless.*

Proof. Let G be a triangle-free graph with $\Delta \geqslant n/2$ with $E(\Lambda[G]) \neq \emptyset$. We shall prove that G is bipartite. Let $xy \in E(\Lambda[G])$. Observe that $N_G(x)$ and $N_G(y)$ are disjoint and each one of them is an independent set, because G is triangle-free. Since $|N_G(x)|$ and $|N_G(y)|$ are both at least $n/2$, we have $N_G(x) \cup N_G(y) = V(G)$, which implies that actually $\Delta = n/2$ and concludes the proof. □

We dedicate the remaining of this section to further theorems which follow from the recolouring procedure presented in Section 3.1. These theorems are stronger statements than the theorems presented in Section 2.2. It should be noticed that each one of the following proofs and the corresponding proof in Section 2.2 are quite similar, with the difference that the former uses the recolouring procedure which we have presented in Section 3.1, whilst the latter uses Vizing's recolouring procedure instead.

The first result is a stronger version of Theorem 2.3 (p. 31). In the statement, the *hard core* of a graph G, denoted $\mathbf{\Lambda}[G]$, is the subgraph of G induced by the majors which have local degree sum at least $\Delta^2 - \Delta + 1$, that is, the majors of G which are proper or tightly non-proper. In the proof, we also use $\mathbb{A}[G]$ to denote the *soft core* of G, that is, the subgraph of G induced by its strictly non-proper majors. Clearly, $\Lambda[G] = \mathbf{\Lambda}[G] \cup \mathbb{A}[G]$.

Theorem 3.14. *Every graph whose hard core is acyclic is* Class 1.

Proof. Let G be a graph with acyclic hard core and maximum degree $\Delta \geqslant 2$. We assume $V(\mathbf{\Lambda}[G]) \neq \emptyset$, since otherwise we would already be done by Theorem 1.10. Remark that $V(\mathbb{A}[G]) \neq \emptyset$ also holds, since the vertices of $V(\mathbf{\Lambda}[G])$ with degree at most one in $\mathbf{\Lambda}[G]$ must be adjacent to at least one vertex in $V(\mathbb{A}[G])$, otherwise they would have local degree sum not greater than $\Delta^2 - \Delta$ in G. Therefore, if we take $G' := G - E(\mathbf{\Lambda}[G])$, then $\Delta(G') = \Delta$ and, since all the majors of G' are strictly non-proper, we can take a Δ-edge-colouring of G' by Theorem 1.10.

Now we shall colour the edges of $\mathbf{\Lambda}[G]$. As in the proof of Theorem 2.3, for each component T (a tree) of $\mathbf{\Lambda}[G]$, we choose an arbitrary vertex to be the *root* of T, defining for all $u \in V(\mathbf{\Lambda}[G])$ the *height* and the *parent* of u in its tree. Recall that, $h(r) = 0$ and $p(r)$ is undefined for every root r. For each $i = 1, \ldots, \max_{u \in V(\mathbf{\Lambda}[G])} h(u)$, consider each $u \in V(\mathbf{\Lambda}[G])$ with $h(u) = i$, one at a time. We shall colour the edge $up(u)$. Let G' be the subgraph of G induced by the edges which have already been coloured by the current moment. Each major w of G adjacent to u in G satisfies one of the following:

1. w is a strictly non-proper major of G;

2. w is in $V(\mathbf{\Lambda}[G])$, but either $w = p(u)$, or w has height greater than $h(u)$, with both cases implying that the edge wu has not been coloured yet and, thus, that w actually misses some colour.

Ergo, we can apply Lemma 3.8 on $up(u)$. □

The following result is a stronger form of Theorem 3.14 when restricted to triangle-free graphs:

Theorem 3.15. *If G is triangle-free and the set of the proper majors of G induces an acyclic graph, then G is* Class 1.

Proof. Analogous to proof for Theorem 3.14, but applying Lemma 3.10 instead of Lemma 3.8 at each edge uv considered. □

Theorem3.16 on the next page highlights the relevance of the result shown in Theorem 3.14. We already know that almost every graph is in $\mathscr{X}$ and thus is covered by

Theorem 1.10. What we show next is that, when we are given that a random graph is not in $\mathscr{X}$, then almost surely its hard core is acyclic, so the graph is covered by Theorem 3.14 (see Figure 3.9).

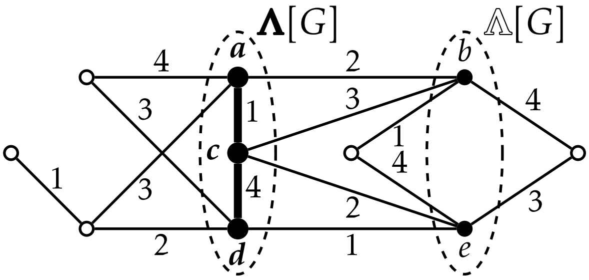


Figure 3.9: An optimal edge-colouring of a *Class 1* graph whose set of majors (filled) can be partitioned into a set which induces an acyclic graph (the hard core, thicker) and a set of strictly non-proper majors (the soft core). Note that the majors in the hard core are proper ($\{c\}$) or tightly non-proper ($\{a, d\}$) and the whole set of majors does not induce an acyclic graph.

Theorem 3.16. *Almost every graph with cycles in the core and* $\Delta > 2$ *which is not in* $\mathscr{X}$ *has acyclic hard core.*

Proof. From Theorem 3.9 follows that if G is a $\mathscr{G}(n, 1/2)$ graph, then almost surely $\mathbb{A}[G] = \Lambda[G]$, even given that $\Lambda[G]$ has a cycle and $\Delta > 2$. Furthermore, if it is also given that $G[X] \neq \Lambda[G]$, the same arguments used in the proof of Theorem 3.9 can be used to verify that almost surely $\Lambda[G] - X$ is unitary, therefore acyclic. □

We know that every vertex of a critical graph is adjacent to at least two majors (Vizing, 1965). With our recolouring procedure, we provide a stronger result:

Theorem 3.17. *Every vertex of a critical graph is adjacent to at least two majors with local degree sum at least* $\Delta^2 - \Delta + 1$.

Proof. We shall first show that every vertex of G is adjacent to at least two vertices of $\mathbf{\Lambda}[G]$. For this it suffices to show that if u is a vertex adjacent to at most one proper or tightly non-proper major in a graph G, and if $G - u$ is *Class 1*, then G is *Class 1*. In order to see that, we first take a Δ-edge-colouring of $G - u$. If all neighbours of u in G are not in $\mathbf{\Lambda}[G]$, then we apply Lemma 3.8 to colour each edge incident to u and we are done. Otherwise, let x be *the* neighbour of u in G such that x is a proper or a tightly non-proper major of G. Again, the proof follows by applying Lemma 3.8 to colour each edge incident to u if we simply leave the edge ux to be coloured at last. □

Recall that the semi-core of a graph G, denoted $\not{\mathrm{A}}[G]$, is the subgraph induced by $\bigcup_{u \in V(\Lambda[G])}(\{u\} \cup N_G(u))$. Analogously, we define the *hard semi-core* of G, denoted $\not{\mathbf{A}}[G]$, which is the subgraph induced by $\bigcup_{u \in V(\mathbf{\Lambda}[G])}(\{u\} \cup N_G(u))$. As the problem of computing an optimal edge-colouring can be reduced to the problem of computing an optimal edge-colouring of its semi-core (Theorem 2.6, p. 35), we show the following stronger statement:

Theorem 3.18. *If the hard core of a graph G has no vertex, then G is* Class 1. *Otherwise, the chromatic index of G is equal to the chromatic index of its hard semi-core.*

Proof. First, recall that graphs with empty hard core are precisely the graphs in $\mathscr{X}$, which we have already proved to be *Class 1* in Theorem 1.10. So, let G be a graph with $V(\mathbf{\Lambda}[G]) \neq \emptyset$, which implies that $V(\mathbf{A}\!\!\!-[G]) \neq \emptyset$ and $\Delta(\mathbf{A}\!\!\!-[G]) = \Delta(G) =: \Delta$. Let also φ be an optimal edge-colouring of $\mathrm{A}\!\!\!-[G]$, being $\mathscr{C}$ the colour set used.

We shall colour the edges of $E(G) \setminus E(\mathbf{A}\!\!\!-[G])$, one at a time, using only colours of $\mathscr{C}$. For each edge uv considered, at least one end-vertex of uv, say u, is not in $V(\mathbf{A}\!\!\!-[G])$. Hence, all neighbours of u in G are either non-majors or strictly non-proper majors of G. Hence, by applying Lemma 3.8 we get a colour of $\mathscr{C}$ to assign to uv. □

3.3 Remarks on time complexity

Theorem 3.19 provides more details on the polynomiality of our recolouring procedure.

Theorem 3.19. *For every graph G in the graph class $\mathscr{X}$, an optimal edge-colouring of G can be computed in $O(\Delta^3 nm)$ time, being $n := |V(G)|$, $m := |E(G)|$, and $\Delta := \Delta(G)$.*

Proof. By the proof of Theorem 1.10, the Δ-edge-colouring of G can be constructed edge by edge applying Lemma 3.8 at each edge uv considered. It suffices then to demonstrate that, for each uv, the overall time complexity of the algorithm implicit in the proof of Lemma 3.8 is $O(\Delta^3 n)$.

First, we show that, being $v_0, \ldots, v_k$ a recolouring fan for uv, testing if this recolouring fan satisfies any amongst Lemmas 3.2–3.7, as well as performing the corresponding constructive proof, can be done in $O(\Delta^2 n)$ time. Assuming that we have an $O(\Delta n)$-space data structure which, for all $u \in V$ and all $\alpha \in \mathscr{C}$, stores which neighbour x of u in $G - uv$ satisfies $\varphi(ux) = \alpha$ (or **nil** if there is none), the following worst-case time complexities can be straightforwardly verified:

Lemma	Testing satisfiability	Performing constructive proof
3.2	$O(\Delta)$	$O(\Delta)$
3.3	$O(\Delta^2 n)$	$O(n)$
3.4	$O(\Delta^2 n)$	$O(n)$
3.5	$O(\Delta)$	$O(\Delta^2 n)$
3.6	$O(\Delta)$	$O(\Delta^2 n)$
3.7	$O(\Delta n)$	$O(\Delta^2 n)$

It should be noticed that, during the construction of the recolouring fan in the proof of Lemma 3.8, the satisfiability of some amongst Lemmas 3.2–3.7 is verified each time a vertex is appended to the fan. Hence, each step of the construction takes $O(\Delta^2 n)$ time, which is not asymptotically inferior to the time complexity of performing the subprocedures established in Cases 1 and 2 of the proof for each time the construction gets stuck. Ergo, since at most Δ vertices are appended to the fan until the colours of the fan decay and the edge uv is coloured, performing the proof of Lemma 3.8 takes $O(\Delta^3 n)$ worst-case time. □

The complexity analysis described in the proof of Theorem 3.19 yields the following corollaries. In all of them, G is a graph, $n := |V(G)|$, $m := |E(G)|$, and $\Delta := \Delta(G)$.

Corollary 3.20. *If the hard core of G is acyclic, then an optimal edge-colouring of G can be computed in $O(\Delta^3 nm)$ time.*

Corollary 3.21. *If u is a vertex adjacent to at most one proper or tightly non-proper major in G, and if $G-u$ is* Class 1 *for which we already have a Δ-edge-colouring, then a Δ-edge-colouring for G can be obtained in $O(d_G(u)\Delta^3 n)$ time.*

Corollary 3.22. *If the hard core of G is not empty and if we already have an optimal edge-colouring of the hard semi-core of G, then an optimal edge-colouring of G can be obtained in $O((m-|E(\mathbf{A}[G])|)\Delta^3 n)$ time.*

4 On edge-colouring complementary prisms and join graphs

In this chapter we present results on edge-colouring complementary prisms and join graphs. In particular, as announced in Chapter 1, we prove that:

THEOREM 1.13. *A complementary prism can be* Class 2 *only if it is a regular graph distinct from the* K_2.

Also, we present evidences for the following conjecture:

CONJECTURE 1.11. *Let* G_1 *and* G_2 *be disjoint graphs such that* $|V(G_1)| \leqslant |V(G_2)|$ *without loss of generality. If* $\Delta(G_1) \geqslant \Delta(G_2)$ *and if the majors of* G_1 *induce an acyclic graph, then the join graph* $G_1 * G_2$ *is* Class 1.

This chapter is organised as follows:

- Section 4.1 presents some preliminary facts on edge-colouring general graphs which shall be useful for the proofs in the sections which follow;
- Section 4.2 briefly presents the state of the art of edge-colouring join graphs and some results of our own which do not lie on an extended recolouring procedure, including a proof for a slightly weaker statement than Conjecture 1.11;
- Section 4.3 presents an extended recolouring procedure which provides further evidences for Conjecture 1.11;
- Section 4.4 presents the proof for Theorem 1.13;
- Section 4.5 closes the chapter discussing a decomposition technique for edge-colouring yielded by the proof of Theorem 1.13, as well as another decomposition technique which leads to a result on chordal graphs.

4.1 Preliminaries for the chapter

We start with a fact on the number of non-majors in a graph with acyclic core.

LEMMA 4.1. *If a graph* G *has maximum degree* $\Delta > 1$ *and an acyclic core with* s *vertices, then* G *has at least*

$$\max\left\{\Delta - 1, s - \left\lfloor \frac{s-2}{\Delta - 1} \right\rfloor\right\}$$

vertices of degree less than Δ.

Proof. Consider the set

$$X := \{(u,e) : u \in V(\Lambda[G]) \text{ and } e \in \partial_G(u)\}.$$

As each $u \in V(\Lambda[G])$ appears in X exactly $d_G(u) = \Delta$ times, we have $|X| = s\Delta$. On the other hand, because

$$\bigcup_{u \in V(\Lambda[G])} \partial_G(u) = E(\Lambda[G]) \cup \partial_G(V(\Lambda[G]))\,,$$

each edge of $E(\Lambda[G])$ appears in X exactly twice, whereas each edge of $\partial_G(V(\Lambda[G]))$ appears in X exactly once, which brings

$$|X| = 2|E(\Lambda[G])| + |\partial_G(V(\Lambda[G]))|$$

and $|\partial_G(V(\Lambda[G]))| \geqslant s(\Delta - 2) + 2$, since $|E(\Lambda[G])| \leqslant s - 1$.

Because for all $u \in V(G) \setminus V(\Lambda[G])$ we have

$$|\partial_G(u) \cap \partial_G(V(\Lambda[G]))| \leqslant s \qquad \text{and}$$
$$|\partial_G(u) \cap \partial_G(V(\Lambda[G]))| \leqslant \Delta - 1\,,$$

and since

$$|\partial_G(V(\Lambda[G]))| = \sum_{u \in V(G) \setminus V(\Lambda[G])} |\partial_G(u) \cap \partial_G(V(\Lambda[G]))|\,,$$

we have both $(n-s)s$ and $(n-s)(\Delta-1)$ at least $s(\Delta-2)+2$, implying

$$n - s \geqslant \left\lceil (\Delta - 2) + \frac{2}{s} \right\rceil \geqslant \Delta - 1 \quad \text{and}$$
$$n - s \geqslant \left\lceil s - \frac{s-2}{\Delta - 1} \right\rceil = s - \left\lfloor \frac{s-2}{\Delta-1} \right\rfloor,$$

since $n - s$ is an integer and $s > 0$. $\square$

The bound in Lemma 4.1 is tight, being the diamond (Figure 1.2, p. 12) an example of a graph with acyclic core and $\Delta - 1 = s - \lfloor (s-2)/(\Delta-1) \rfloor = 2$ non-majors.

The following is a standard result implied by Theorem 2.8 (p. 36).

Observation 4.2. *Every n-vertex graph G with maximum degree Δ has a $(\Delta+1)$-edge-colouring in which at least* $\min\{\Delta+1, \lceil n/2 \rceil\}$ *colours are missed by at least one vertex each.*

Proof. In an equitable $(\Delta+1)$-edge-colouring of G, whose existence is guaranteed by Theorem 2.8, every colour is assigned to either $\lfloor m/(\Delta+1) \rfloor$ or $\lceil m/(\Delta+1) \rceil$ edges and, thus, is *not* missing at either $2\lfloor m/(\Delta+1) \rfloor$ or $2\lceil m/(\Delta+1) \rceil$ vertices. Hence, if in such an edge-colouring the number of colours missing at no vertex is $k = 0$, we already have $\Delta+1$ colours which are missed by at least one vertex each, so nothing remains to be shown. However, if $k > 0$, then $\Delta + 1 - k$ colours are missing at 2 vertices each. As each vertex misses at least one colour, this brings $2(\Delta + 1 - k) \geqslant n$ and hence $\Delta + 1 - k \geqslant \lceil n/2 \rceil$. $\square$

Lemma 4.3 below is a classical result on edge-colouring regular graphs which is often referred to as the *Parity Lemma* in the literature. The proof for this lemma was originally presented by Isaacs (1975) for cubic graphs, but it actually works for d-regular graphs for all $d \geqslant 2$, as shown.

Lemma 4.3 (The Parity Lemma (Isaacs, 1975)). *Let G be a* Class 1 *d-regular graph with $d \geqslant 2$ and let F be a* cut *in G. Let also α and β be two colours used in a d-edge-colouring of G, being f_α and f_β the numbers of edges in F coloured α and coloured β, respectively. Then*

$$f_\alpha \equiv f_\beta \pmod 2.$$

Proof. Since no colour is missing at any vertex in the d-edge-colouring, we know that the subgraph of G induced by the edges coloured α or β is a disjoint union of even cycles $C_1,\dots,C_k$ for some $k \geqslant 1$. Hence, $|E(C_i) \cap F|$ must be even for each $i \in \{1,\dots,k\}$, since F is a cut. Therefore, in each one of these cycles the parity of the number $f_\alpha^{(i)}$ of α-coloured edges in $E(C_i) \cap F$ must be equal to the parity of the number $f_\beta^{(i)}$ of β-coloured edges in $E(C_i) \cap F$. The proof is concluded by observing that $f_\alpha = \sum_{i=1}^{k} f_\alpha^{(i)}$ and $f_\beta = \sum_{i=1}^{k} f_\beta^{(i)}$. □

From the Parity Lemma follow interesting facts on edge-colouring regular graphs, such as the standard results stated in Observations 4.4 and 4.5.

Observation 4.4. *Every d-regular graph G with a bridge is* Class 2, *for any $d \geqslant 2$.*

Proof. If G is a d-regular graph with a bridge e and a d-edge-colouring φ, then the parity of the number of edges in $\{e\}$ coloured α is 1, if $\alpha = \varphi(e)$, or 0, otherwise. Since $\{e\}$ is a cut, this contradicts the Parity Lemma (Lemma 4.3). □

Observation 4.5. *If d is an odd integer and G is a* Class 2 *d-regular graph on an even number of vertices, then $G-u$ is also* Class 2 *for any $u \in V(G)$.*

Proof. We shall use the Parity Lemma (Lemma 4.3) to show the contraposition of the statement. Let d be an odd integer, let G be a d-regular graph such that $G-u$ is *Class 1*, and take any d-edge-colouring of $G-u$. We shall demonstrate how to construct a d-edge-colouring of G.

From the graph $G-u$ and the d-edge-colouring taken, we create the d-edge-colourable graph G' as follows: for each neighbour v of u in G, we add a new pendant vertex v' to $G-uv$, connecting it only with v and assigning to the edge vv' the only colour which is missing at v. Let F be the set of the created edges. Since F is a cut in G' with exactly k edges, and since k is odd, the lemma brings that each one of the k colours is assigned to exactly one of the edges in F. The proof is concluded by identifying all the k new vertices to the vertex u. □

Remark that Observation 4.5 does not hold when d is even, being the K_3 the smallest counterexample of a critical regular graph.

We close this section with an interesting application of the Pigeonhole Principle.

Lemma 4.6 (joint with A. Zorzi). *Let $G = (V,E)$ be an n-vertex graph with $m \leqslant \lceil \Delta(n-1)/2 \rceil$ edges and maximum degree Δ. Then, for all $k \in \{m,\dots,\lceil \Delta(n-1)/2 \rceil\}$, there is a multigraph $\mathcal{G}$ with $|E(\mathcal{G})| = k$, $\Delta(\mathcal{G}) = \Delta$, and $\chi'(\mathcal{G}) \leqslant \Delta + 1$ which has G as a spanning subgraph.*

Proof. If $k = m$, it suffices to take $\mathcal{G} := G$. Otherwise, consider a $(\Delta+1)$-edge-colouring of G and let R be the set of all pairs $(\{u,v\},\alpha)$ such that u and v are any vertices of G and α is a colour missing at both. We must have

$$\sum_{u \in V} (\Delta - d_G(u)) \geqslant \Delta + 2(k-m), \tag{4.1}$$

otherwise

$$\Delta n - \sum_{u \in V} d_G(u) \geqslant \Delta + 2k - 2m$$

and, since $\sum_{u \in V} d_G(u) = 2m$,

$$\Delta n - 2k \leqslant \Delta - 1,$$

which implies

$$2k \geqslant \Delta(n-1)+1$$

and, since k is an integer,

$$k \geqslant \left\lceil \frac{\Delta(n-1)+1}{2} \right\rceil > \left\lceil \frac{\Delta(n-1)}{2} \right\rceil,$$

a contradiction.

Due to (4.1), we claim that we can take some $S \subseteq R$ such that $|S| = k - m$ and no vertex u appears in more than $\Delta - d_{\mathcal{G}}(u)$ of the pairs in S. Therefore, all we have to do is to add a (possibly multiple) edge uv to $\mathcal{G}$ and colour it α, for any such $(\{u,v\},\alpha) \in S$. In order to prove the claim, observe by the Pigeonhole Principle that there must be some $(\{u,v\},\alpha) \in R$ with both $d_{\mathcal{G}}(u)$ and $d_{\mathcal{G}}(v)$ strictly less than Δ, since we have only $\Delta + 1$ colours. If we add the edge uv to $\mathcal{G}$ and remove it from R, we decrease the sum $\sum_{u \in V}(\Delta - d_{\mathcal{G}}(u))$ exactly by 2. If $k - m > 1$, there is again some $(\{u',v'\},\alpha) \in R$ with both $d_{\mathcal{G}}(u')$ and $d_{\mathcal{G}}(v')$ less than Δ, so we repeat the procedure another $k - m - 1$ times. □

4.2 An introduction to edge-colouring join graphs

Before we start, we remark that some authors, when defining a join graph $G_1 * G_2$, do not assume that G_1 and G_2 are disjoint graphs, differently from our definition of join graphs presented in Chapter 1. The two definitions are equivalent, except for the particular case of the K_1. If we do not assume disjointness, the K_1 can be considered a join graph (since it could be given by the join of two complete graphs on the same unitary vertex set); otherwise, it cannot be a join graph. Excluding the particular case of the K_1, which makes no difference in the context of edge-colouring join graphs, the operands G_1 and G_2 can always be assumed disjoint because, if they are not, we can simply replace G_1 and G_2 with the graphs G_1' and G_2', respectively, defined by:

$$V(G_1') = V(G_1); \qquad E(G_1') = E(G_1) \cup \{uv : u,v \in V(G_1) \cap V(G_2)\};$$
$$V(G_2') = V(G_2) \setminus V(G_1); \qquad E(G_2') = E(G_2) \setminus \{uv \in E(G_2) : u \in V(G_1)\}.$$

Remark that this argument can be used to show that a graph distinct from the K_1 is a join graph if and only if its complement is disconnected. Hence, join graphs can be recognised in linear time with the algorithms presented by Ito and Yokoyama (1998).

We have presented in Chapter 2 many graph classes for which optimal vertex-colourings can be computed in polynomial time, but for which the complexity of CHRIND is $\mathcal{NP}$-complete or it is still undetermined. Join graphs seem to *break* this phenomenon in some sense. Since all n-vertex join graphs satisfy $\Delta \geqslant n/2$, the Overfull Conjecture suggests the existence of a linear-time algorithm for CHRIND when restricted to join graphs, as discussed in Chapter 2. On the other hand, computing optimal vertex-colourings is clearly $\mathcal{NP}$-hard even for graphs with a universal vertex (a subclass of join graphs for which edge-colourings can be computed in polynomial time according to Plantholt (1981), recall Table 1.1 (p. 22)).

Besides graphs with a universal vertex, other subclasses of join graphs for which the Overfull Conjecture has already been proved are the complete multipartite graphs (Hoffman and Rodger, 1992) and the regular join graphs (De Simone and Galluccio, 2007). For the remaining of the class, only partial results are known, mostly sufficient conditions for a join graph to be *Class 1*. Theorem 4.7 presents some of these conditions which we have found in the literature. In the statement, and throughout this text, we define, for $i \in \{1,2\}$, $n_i := |V_i|$, $m_i := |E_i|$, and $\Delta_i := \Delta(G_i)$.

Theorem 4.7 (De Simone and Mello, 2006; Machado and Figueiredo, 2010; Cunha Lima et al., 2015). *The following conditions are sufficient for a join graph $G = G_1 * G_2$ with $n_1 \leqslant n_2$ to be* Class 1*:*

(1) $\Delta_1 > \Delta_2$ *(De Simone and Mello, 2006);*

(2) $n_2 - n_1 > \Delta_2$ *(Machado and Figueiredo, 2010);*

(3) G_1 *and* G_2 *are regular graphs and* $n_1 + n_2$ *is even (De Simone and Galluccio, 2009, 2013).*

(4) $\Delta_1 = \Delta_2 =: d$ *and at least one of the following holds:*

   (a) *both* G_1 *and* G_2 *are* Class 1 *De Simone and Mello (2006);*

   (b) *each connected component of* $G_1 \cup G_2$ *has at most* $d+1$ *vertices (De Simone and Mello, 2006; Cunha Lima et al., 2015);*

   (c) *each connected component of* G_1 *has at most* $d+1$ *vertices,* G_2 *is bipartite, and either* d *is odd or* G_2 *has no induced* $K_{d,d}$ *(Cunha Lima et al., 2015);*

   (d) $\Lambda[G_1]$ *is edgeless (Cunha Lima et al., 2015).*

The results in Theorem 4.7(4b–4c) were stated by the authors under $n_1 = n_2$, but we remark that the same proofs provided work with $n_1 < n_2$.

Throughout the remaining of this chapter, when dealing with a join graph $G = G_1 * G_2$, we always assume $n_1 \leqslant n_2$ without loss of generality. We present results for the case wherein $\Delta_1 \geqslant \Delta_2$, which implies $\Delta(G) = \Delta_1 + n_2$. In this context, we define $B_G := G - (E_1 \cup E_2)$, which is a complete bipartite subgraph of G. Observe that every maximal matching M on B_G covers $V(G_1)$ and, if $n_1 = n_2$, then M is actually a perfect matching on B_G. Since we know that G is *Class 1* if $\Delta_1 > \Delta_2$ (Theorem 4.7(1)), we assume in all the proofs (not in the statements) in this chapter, except for the proof of Lemma 4.17 in p. 74, that $\Delta_1 = \Delta_2 =: d$.

Let $G_M := (G_1 \cup G_2) + M$ for any maximal matching M on B_G and $M(x)$ be the vertex matched to x by M for any $x \in V(G_1)$ (see Figure 4.1).

The proofs of Theorem 4.7(1 and 4) follow from an observation by De Simone and Mello (2006) which provides a decomposition technique for edge-colouring join graphs with $\Delta_1 \geqslant \Delta_2$. This technique shall be used to prove some of our results.

Observation 4.8 (De Simone and Mello, 2006). *If* $\Delta_1 \geqslant \Delta_2$ *and* G_M *is* Class 1, *then* G *is also* Class 1.

Proof. Follows from the facts that: $G = G_M \cup (B_G - M)$, $\Delta(G_M) = \Delta_1 + 1$, and $B_G - M$ is a bipartite, hence *Class 1* (Theorem 1.3, p. 18), graph satisfying $\Delta(B_G - M) = n_2 - 1$. Ergo, in order to obtain a $\Delta(G)$-edge-colouring for G, it suffices to take a $(\Delta+1)$-edge-colouring of G_M and a (n_2-1)-edge-colouring of $B_G - M$ on disjoint sets of colours. □

Next, using Vizing's original recolouring procedure, we present a proof for a weaker statement than Conjecture 1.11:

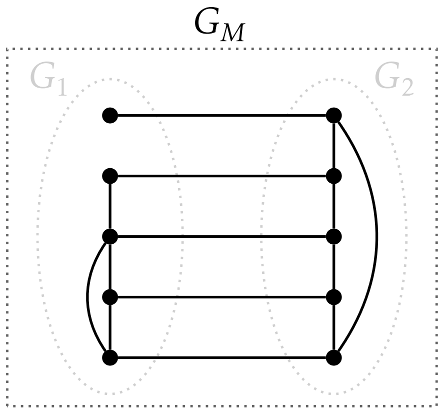


Figure 4.1: The graph G_M corresponding to a perfect matching on the graph B_G for the join graph of Figure 1.17

Theorem 4.9. *If $\Delta_1 \geqslant \Delta_2$ and $\Lambda[G_1]$ is acyclic, and if, being $T_1, \dots, T_k$ the connected components of $\Lambda[G_1]$,*

$$|\{u \in V(\Lambda[G_1]) : d_{\Lambda[G_1]}(u) > 1\}| + |\{T_i : |V(T_i)| = 2\}| \leqslant n_2 - |V(\Lambda[G_2])|,$$

then G is Class 1.

Proof. First note that if $n_1 < n_2$, then $\Lambda[G] = \Lambda[G_1]$ is acyclic, and thus G is *Class 1* by Theorem 2.3 (p. 31). So, the only remaining case is when $n_1 = n_2$ and, by consequence, $\Delta(G) = n_1 + d = n_2 + d$.

For each component T_i of $\Lambda[G_1]$, observe that T_i is a tree and: if T_i has more than two vertices, we pick up a vertex with degree in T_i greater than one to be the *root* of T_i; otherwise, we pick up any vertex of T_i to be the *root* of T_i. Being r_i the chosen root of T_i, we define for each $u \in V(T_i)$ the *height* of u, denoted $h(u)$, the *parent* of u, denoted $p(u)$, and the *children* of u as all the neighbours of u distinct from $p(u)$. Furthermore, we define $h(T_i) := \max_{u \in V(T_i)} h(u)$ and the *leaves* of T_i as the vertices of $T_i - r_i$ without children. Notice that, if T_i has only two vertices, one shall be considered the root and the other shall be considered a leaf. Also, if T_i is trivial, then its unique vertex shall be considered a root, not a leaf.

Let M be a perfect matching on B_G such that every $u \in V(\Lambda[G_1])$ which is not a leaf and does not belong to a trivial T_i has at least one child w satisfying $M(w) \notin V(\Lambda[G_2])$. Because a vertex $u \in V(\Lambda[G_1])$ is not a leaf and does not belong to a trivial T_i if and only if $d_{\Lambda[G_1]}(u) > 1$ or $u = r_i$ for some T_i with two vertices, the choice of the matching M does not require more than

$$|\{u \in V(\Lambda[G_1]) : d_{\Lambda[G_1]}(u) > 1\}| + |\{T_i : |V(T_i)| = 2\}|$$

non-majors in G_2, whose existence are guaranteed by hypothesis.

We shall colour the edges of G_M using a set $\mathscr{C}$ of $d + 1$ colours by performing the following algorithm:

Step 1. For each $u \in V(\Lambda[G_1])$ which is not a leaf and belongs to a non-trivial T_i, choose one child w of u satisfying $M(w) \notin V(\Lambda[G_2])$ and put the edge uw in an edge set F, initially empty. Then, as $\Delta((G_1 - F) \cup G_2) = d$, and thus $\chi'((G_1 - F) \cup G_2) \leqslant d + 1$, take any edge-colouring of $(G_1 - F) \cup G_2$ with $\mathscr{C}$.

Step 2. Consider each edge uv of M such that $u \in V(G_1) \setminus V(\Lambda[G_1])$. For each neighbour w of u in G_M, we have one of the following cases:

(i) $w = v$, in which case w misses a colour of $\mathscr{C}$ because uv is the edge that we are about to colour;

(ii) $w \in V(G_1) \setminus V(\Lambda[G_1])$, in which case w misses a colour of $\mathscr{C}$ because $d_{G_M}(w) < d + 1$;

(iii) $w \in V(\Lambda[G_1])$, in which case w misses a colour because no edge of M incident to a vertex of $\Lambda[G_1]$ has been coloured yet.

Since uv satisfies the conditions of Lemma 1.9, let it receive a colour of $\mathscr{C}$.

Step 3. Then, consider each $u \in V(\Lambda[G_1])$ with $h(u) = 0$. For each neighbour w of u in G_M, we have one of the following cases:

(i) w is $M(u)$, in which case w misses a colour of $\mathscr{C}$ because $uM(u)$ is the edge we are about to colour;

(ii) $w \in V(G_1) \setminus V(\Lambda[G_1])$, in which case w clearly misses a colour of $\mathscr{C}$;

(iii) $w \in V(\Lambda[G_1])$ and $h(w) > 0$, in which case w misses a colour because no edge of M incident to vertices of $\Lambda[G_1]$ with height greater than zero has been coloured yet.

Ergo, $uM(u)$ satisfies the condition of Lemma 1.9 and receives a colour of $\mathscr{C}$.

Step 4. For h from 1 to $\max_{i=1}^{k} h(T_i)$, perform:

Step 4.1. Consider each $u \in V(\Lambda[G_1])$ with $h(u) = h$. For each neighbour w of u in G_M, we have one of the following cases:

(i) $w = M(u)$;

(ii) $w \in V(G_1) \setminus V(\Lambda[G_1])$;

(iii) $w \in V(\Lambda[G_1])$, in which case w misses a colour because either $h(w) > h$ and the edge $wM(w)$ has not been coloured yet, or $w = p(u)$ and, although the edge $wM(w)$ is already coloured, we know that w has a child v (possibly u, but not necessarily) such that the edge wv is in F and thus has not been coloured yet.

Therefore, the condition of Lemma 1.9 is satisfied by $uM(u)$.

Step 4.2. Now that all edges of M incident to vertices of $\Lambda[G_1]$ with height h have received a colour, we can consider each edge $up(u) \in F$ such that $h(u) = h$. For each neighbour w of u in G_M all the possibilities are:

(i) $w = p(u)$, in which case w misses a colour because $up(u)$ is the edge we are about to colour;

(ii) $w \in V(G_1) \setminus V(\Lambda[G_1])$;

(iii) $w = M(u)$, in which case w misses a colour because, by the manner we have built the set F, $M(u) \notin V(\Lambda[G_2])$;

(iv) $h(w) > h$, in which case w misses a colour because the edge $wM(w)$ has not been coloured yet.

Hence, we also apply Lemma 1.9 to colour $up(u)$.

The proof is concluded by Observation 4.8. □

Next, we present an interesting counting argument for edge-colouring joins of graphs with same order and same number of edges.

Theorem 4.10 (joint with A. Zorzi). *If $n_1 = n_2$, $\Delta_1 \geqslant \Delta_2$, and $m_1 = m_2$, then G is* Class 1.

Proof. Let $\mathscr{C} = \{1, \ldots, d+1\}$ and take an equitable edge-colouring for G_1 and an equitable edge-colouring for G_2, with $\mathscr{C}$ as the colour set of both edge-colourings (recall Theorem 2.8, p. 36). For each colour $\alpha \in \mathscr{C}$ and each $i \in \{1,2\}$, let $m_{G_i}(\alpha)$ be the number of α-coloured edges in the graph G_i. We can assume

$$m_{G_1}(1) \leqslant \cdots \leqslant m_{G_1}(d+1)$$

and

$$m_{G_2}(1) \leqslant \cdots \leqslant m_{G_2}(d+1),$$

without loss of generality. Additionally, by the equitability of each edge-colouring taken, if $d+1$ divides $m_1(= m_2)$, then

$$m_{G_1}(\alpha) = m_{G_2}(\alpha) = \frac{m_1}{d+1} \qquad \forall \alpha \in \mathscr{C}.$$

But, if $d+1$ does not divide m_1, it can also be demonstrated that $m_{G_1}(\alpha) = m_{G_2}(\alpha)$ for every $\alpha \in \mathscr{C}$: if we take the colour i_1 of $\mathscr{C}$ such that $m_{G_1}(j) = \lfloor m/(d+1) \rfloor$ for every $j \leqslant i_1$ and $m_{G_1}(j) = \lceil m/(d+1) \rceil$ for every $j > i_1$, and if we also take the colour i_2 of $\mathscr{C}$ such that $m_{G_2}(j) = \lfloor m/(d+1) \rfloor$ for every $j \leqslant i_2$ and $m_{G_2}(j) = \lceil m/(d+1) \rceil$ for every $j > i_2$, then we must have $i_1 = i_2$, otherwise m_1 and m_2 could not be equal.

Now, if we denote by $\overline{n}_{G_i}(\alpha)$ the number of vertices of the graph G_i which miss the colour α, we can verify for all $\alpha \in \mathscr{C}$ that

$$\overline{n}_{G_1}(\alpha) = n - 2m_{G_1}(\alpha) = n - 2m_{G_2}(\alpha) = \overline{n}_{G_2}(\alpha).$$

Therefore, we can take a perfect matching M on B_G such that, for each edge $uv \in M$, there is a colour $\alpha \in \mathscr{C}$ missing at both u and v, which we can use to colour uv and thus obtain a $\Delta(G_M)$-edge-colouring of G_M. By Observation 4.8, the proof is concluded. □

We remark that the proof of Theorem 2.8 (p. 36) holds even if the graph has multiple edges. So, the proof for Theorem 4.10 can also be used to prove that $\chi'(G) = \Delta(G)$ even in the case wherein $G_1 \cup G_2$ has multiple edges, as long as $\chi'(G_1 \cup G_2) \leqslant \Delta_1 + 1$.

Still concerning Theorem 4.10, sometimes $m_1 \neq m_2$ but we can create new edges in one of the graphs without modifying its maximum degree in order to get equal number of edges. This is what is stated in Corollary 4.11.

Corollary 4.11 (joint with A. Zorzi). *If $n_1 = n_2$, $\Delta_1 \geqslant \Delta_2$, $m_1 \leqslant m_2$, and G_1 is a spanning subgraph of a multigraph $\mathcal{G}_1$ which satisfies $\Delta(\mathcal{G}_1) = \Delta_1$, $\chi'(\mathcal{G}_1) \leqslant \Delta_1 + 1$, and $|E(\mathcal{G}_1)| = m_2$, then G is* Class 1.

Please note the important role of multiple edges in Corollary 4.11. By instance, the graphs $G_1 := K_3 \cup K_2$ and $G_2 := C_5$ satisfy the conditions of Corollary 4.11, but the only edge that we can create in G_1 without increasing its maximum degree is another edge between the vertices from the K_2. An example of a join graph with $n_1 = n_2$ and $\Delta_1 = \Delta_2$ in which such edges cannot be created, even when allowing multiple edges, is the graph G defined by $G_1 := (K_3 \cup K_1)$ and $G_2 := C_4$, which happens to be *Class 2* (since $K_3 * C_4$ is an overfull $\Delta(G)$-subgraph of G). However, the converse of Corollary 4.11 does not hold, as one might suspect. As a counterexample, take $G_1 := K_7 \cup \overline{K_3}$ and G_2 as any 6-regular 10-vertex graph. In this case we cannot create 9 edges in G_1 without increasing its maximum degree or its chromatic index (the only possible 9 edges which can be added to G_1 without increasing its maximum degree would transform the $\overline{K_3}$ in the Shannon multigraph). Nevertheless, $G_1 * G_2$ is *Class 1*:

Observation 4.12 (joint with A. Zorzi). *Being $G_1 := K_7 \cup \overline{K_3}$ and G_2 any 6-regular 10-vertex graph, the join graph $G := G_1 * G_2$ is* Class 1.

Proof. Let x be any one of the three isolated vertices of G_1. In the graph $G' := G - x$, which satisfies $\Delta(G') = \Delta(G) = 16 =: \Delta$, the only majors are the vertices of the K_7 in G_1, which implies that $\cancel{A}[G] = K_7 * G_2$. In $\cancel{A}[G]$, all the vertices of the K_7 have local degree sum $226 \leqslant \Delta^2 - \Delta = 240$. Ergo, $\cancel{A}[G]$ is *Class 1* by Theorem 2.6 (p. 35) and hence G' is *Class 1* by Theorem 1.10 (p. 24). By the way, the Δ-edge-colourability of $\cancel{A}[G]$ also follows from Theorem 4.18, which shall be presented in p. 75 and whose proof yields a more efficient algorithm than the proof of Theorem 1.10.

Now it remains to colour the edges incident to x. Observe that, in G, all the vertices of G_2 are majors of G with local degree sum 238, that is, strictly non-proper majors of G. Therefore, each one of the edges incident to x can be coloured applying Lemma 3.8 (p. 55). □

The following result is another consequence of Theorem 4.10:

Theorem 4.13 (joint with A. Zorzi). *If $\Delta_1 \geqslant \Delta_2$ and $m_1 \leqslant m_2 \leqslant \lceil \Delta_1(n_1 - 1)/2 \rceil$, then G is* Class 1.

Proof. We assume $n_1 = n_2$ (adding $n_2 - n_1$ isolated vertices to G_1 if necessary). Using Lemma 4.6, we take a $(\Delta_1 + 1)$-edge-colourable multigraph $\mathcal{G}$ with m_2 edges and G_1 as a subgraph. The rest of the proof follows from Theorem 4.10. □

Corollary 4.14 (joint with A. Zorzi). *If $n_1 < n_2$, $\Delta_1 \geqslant \Delta_2$, and $m_1 \geqslant m_2$, then G is* Class 1.

Proof. Follows immediately from adding $n_2 - n_1$ isolated vertices to G_1 and interchanging the roles of G_1 and G_2 in Theorem 4.13. □

Theorem 4.15 below characterises the only possible overfull $\Delta(G)$-subgraph of a join graph G with $\Delta_1 \geqslant \Delta_2$.

Theorem 4.15 (joint with A. Zorzi). *If $\Delta_1 \geqslant \Delta_2$, then the join graph G has an induced overfull $\Delta(G)$-subgraph H if and only if:*

- *either $n_2 - n_1 = 1$ and $H = G$,*
- *or $n_1 = n_2$, the graph G has a unique vertex x of minimum degree, and $H = G - x$.*

Proof. Let H be an induced overfull $\Delta(G)$-subgraph of G. Inasmuch as G is *Class 1* if $\Delta_1 > \Delta_2$ (Theorem 4.7(1)), hence not *SO*, we must have $\Delta_1 = \Delta_2$. Also, by Observation 2.17 (p. 43), we know that $\sum_{u \in V(H)}(\Delta(H) - d_H(u)) \leqslant \Delta(H) - 2$, that $|V(H)|$ is odd, and that H is unique because every n-vertex general graph with maximum degree $\Delta \geqslant n/2$ has at most one induced overfull Δ-subgraph. It suffices to prove:

(a) $H = G$ or $H = G - x$ for some $x \in V(G)$, if $n_1 + n_2$ is odd or even, respectively;

(b) $n_2 - n_1 \leqslant 1$.

If (a) is false, the set $U := V(G) \setminus V(H)$ has least 2 vertices if $n_1 + n_2$ is odd, or 3 otherwise. Moreover, all such vertices must be either in V_1 or V_2, the latter being possible only if $n_1 = n_2$. So, we assume $|U| \geqslant 2$ and $U \subseteq V_1$, which implies $V(\Lambda[H]) \subseteq V(\Lambda[G_1])$. As $n_2 \geqslant n_1 > \Delta_1$, we get a contradiction:

$$\sum_{u \in V(H)} (\Delta(H) - d_H(u)) \geqslant \sum_{u \in V_2} (\Delta(G) - d_G(u) + 2) \geqslant 2n_2 > \Delta(H) - 2. \tag{4.2}$$

Now assume (b) is false. If $n_1 + n_2$ is odd, we know by (a) that $H = G$. As $n_2 - n_1 \geqslant 2$, we can again derive the same contradiction in (4.2). On the other hand, if $n_1 + n_2$ is even, we know by (a) that $H = G - x$ for some unique $x \in V_1$. Replacing G_1 by H in the case wherein $n_1 + n_2$ is odd concludes the proof. □

Remark that Theorem 4.15 provides an evidence for Conjecture 1.11:

Corollary 4.16. *Every join graph $G = G_1 * G_2$ with $n_1 \leqslant n_2$, $\Delta_1 \geqslant \Delta_2$, and $\Lambda[G_1]$ acyclic is not* SO.

Proof. If $n_1 < n_2$, then $\Lambda[G] = \Lambda[G_1]$ and G is *Class 1* by Theorem 2.3 (p. 31), hence not *SO*. If $n_1 = n_2$ and if we assume for the sake of contradiction that G is *SO*, then there is a unique vertex $x \in V(G)$ such that $G - x$ is overfull. We know that x cannot be in G_1, otherwise we would be back to the case wherein $n_1 < n_2$, for which we already know that G is not *SO*. Therefore, $x \in V(G_2)$, but then we have by Lemma 4.1 that

$$\sum_{u \in V(G-x)} (\Delta - d_{G-x}(u)) \geqslant n_1 + d - 1 = \Delta(G) - 1$$

which contradicts Observation 2.17 (p. 43). □

Next, Lemma 4.17 presents a novel decomposition technique for edge-colouring join graphs. Theorem 4.18 uses this technique in order to extend Theorem 4.7(2) when restricted to $\Delta_1 \geqslant \Delta_2$.

Lemma 4.17 (joint with A. Zorzi). *If $n_1 < n_2$, $\Delta_1 \geqslant \Delta_2$, and there are $F \subseteq E(B_G)$ and $R \subseteq E_2$ such that all $u \in V_1$ satisfy*

$$d_{G_1}(u) + 1 \leqslant d_{G_1}(u) + d_{B_G[F]}(u) \leqslant \Delta_1 + 1\,,$$

all $u \in V_2$ satisfy

$$d_{G_2}(u) - (n_2 - n_1 - 1) \leqslant d_{G_2 - R}(u) + d_{B_G[F]}(u) \leqslant \Delta_1 + 1\,,$$

and both $G_{F,R} := (G_1 \cup (G_2 - R)) + F$ and $G_{\overline{F,R}} := B_G - F + R$ are Class 1, *then G is* Class 1.

Proof. It can be verified that the conditions set in the statement imply $G = G_{F,R} \cup G_{\overline{F,R}}$,

$\Delta(G_{F,R}) = \Delta_1 + 1$ and $\Delta(G_{\overline{F,R}}) = n_2 - 1$, because

$$d_{G_{F,R}}(u) = \begin{cases} d_{G_1}(u) + d_{B_G[F]}(u) = \Delta_1 + 1\,, & \text{if } u \in V(\Lambda[G_1]);\\ d_{G_1}(u) + d_{B_G[F]}(u) \leqslant \Delta_1 + 1\,, & \text{if } u \in V_1 \setminus V(\Lambda[G_1]);\\ d_{G_2 - R}(u) + d_{B_G[F]}(u) \leqslant \Delta_1 + 1\,, & \text{if } u \in V_2; \end{cases}$$

$$d_{G_{\overline{F,R}}}(u) = \begin{cases} n_2 - d_{B_G[F]}(u) = n_2 - 1\,, & \text{if } u \in V(\Lambda[G_1]);\\ n_2 - d_{B_G[F]}(u) \leqslant n_2 - 1\,, & \text{if } u \in V_1 \setminus V(\Lambda[G_1]);\\ n_1 - \overbrace{d_{B_G[F]}(u)}^{\geqslant d_{G_2[R]}(u) - (n_2 - n_1 - 1)} + d_{G_2[R]}(u) \leqslant n_2 - 1\,, & \text{if } u \in V_2. \end{cases}$$

We obtain a $\Delta(G)$-edge-colouring of G by using two disjoint sets: one with $\Delta_1 + 1$ colours in order to colour $G_{F,R}$, and the other with $n_2 - 1$ colours in order to colour $G_{\overline{F,R}}$. □

Recall that if $n_2 - n_1 \geqslant 2$, then G cannot be *SO* (Theorem 4.15). We show below that these graphs are actually *Class 1*.

THEOREM 4.18 (joint with A. Zorzi). *If $n_2 - n_1 \geqslant 2$ and $\Delta_1 \geqslant \Delta_2$, then G is* Class 1.

Proof. We start with F and R empty and, due to Observation 4.2, a $(d+1)$-edge-colouring of $G_{F,R} = G_1 \cup G_2$ using a colour set $\mathscr{C}'$ such that $|\mathscr{C}'^{*}| \geqslant \lceil (d+1)/2 \rceil$, wherein $\mathscr{C}'^{*}$ is the set of all $\alpha \in \mathscr{C}'$ missed by at least one vertex in G_1 and one in G_2. Initially, all $u \in V_1 \cup V_2$ satisfies

$$d_{G_{F,R}}(u) \leqslant d + 1 \tag{4.3}$$

and all $u \in V_2$ satisfies

$$d_{G_2[R]}(u) - d_{B_G[F]}(u) \leqslant n_2 - n_1 - 2\,. \tag{4.4}$$

We shall make $G_{F,R}$ fulfil all the requirements of Lemma 4.17. Starting with $U := V_1$, we remove vertices from U until $U = \emptyset$. For each u removed, we choose an edge $uv \in B_G$ to be added to F, assigning to it a colour of $\mathscr{C}'$ and possibly choosing some $vw \in E_2$ to be uncoloured and added to R. After each such operation, it should be observed that all the vertices in V_1 still satisfy (4.3) and all in V_2 still satisfy (4.3) and (4.4). The whole process is done in three phases, described below. Note that the second and the third phases may not even occur, since we only go to the next phase when the condition for the current one does not hold any more. Note also that each $u \in U$ always misses a colour of $\mathscr{C}'$.

Phase 1. There is a colour $\alpha \in \mathscr{C}'$ missing at some $u \in U$ and at some $v \in V_2$.

As v satisfies (4.3) strictly, we remove u from U and add uv to F, assigning α to uv. Observe that this decreases by one the value of $d_{G_2[R]}(v) - d_{B_G[F]}(v)$.

Phase 2. There is a colour $\alpha \in \mathscr{C}'$ missing at two vertices u and u' in U.

Since α is not missing at any vertex in V_2 and $|F| < n_2$, there must be an edge $vw \in E_2 \setminus R$ coloured α. Then, we remove u and u' from U, add uv and $u'w$ to F, assigning to both the colour α, and add vw to R, uncolouring it. Note that this does not change the value of $d_{G_2[R]}(x) - d_{B_G[F]}(x)$ for any $x \in V_2$.

Presets for Phase 3. If Phase 2 is over and $U \neq \emptyset$, we choose for each $\alpha \in \mathscr{C}'^*$ some $v_\alpha \in V_2$ which received an edge of F coloured α in Phase 1. If there is none, we choose any $v_\alpha \in V_2$ which missed α before Phase 1. This latter case is possible only if some $u \in V_1$ which missed α received an edge of F coloured $\beta \in \mathscr{C}'^* \setminus \{\alpha\}$. As both u and v_α still miss α, we add the extra edge uv_α to F and colour it α. Now, for all $\alpha \in \mathscr{C}'^*$, let $\mu(v_\alpha)$ denote the number of occurrences of v_α in the pairs of the set $X = \{(v_\beta, \beta) : \beta \in \mathscr{C}'^*\}$. Then $d_{G_2[R]}(v_\alpha) - d_{B_G[F]}(v_\alpha) \leqslant n_2 - n_1 - 2 - \mu(v_\alpha)$. We claim that $|U| < \lceil (d+1)/2 \rceil \leqslant |\mathscr{C}'^*| = |X|$. In order to see that, let $k := |\{v \in V_2 : d_{G_{F,R}}(v) < d+1\}|$. It is clear that $k \geqslant |U| + 2$, since $n_2 - n_1 \geqslant 2$ and in Phases 1 and 2 we add exactly one edge to F by each vertex removed from U. On the other hand, as Phase 2 is over and $U \neq \emptyset$, no colour of $\mathscr{C}'$ is missing at more than one vertex in $U \cup V_2$, so $k + |U| \leqslant d + 1$. Therefore, $|U| + 2 \leqslant d + 1 - |U|$, which implies $|U| \leqslant (d + 1 - 2)/2$.

Phase 3. No colour of $\mathscr{C}'$ is missing at more than one vertex in $U \cup V_2$.

Remove any u from U and any (v_β, β) from X. There are some $\alpha \in \mathscr{C}'$ missing at u and some edge e incident to v_β coloured α. We have two cases:

1. If $e = v_\beta w \in E_2 \setminus R$, then add the edge uw to F, colouring it α, and add e to R, uncolouring it.

2. If $e = u'v_\beta \in F$, we claim that there is some $vw \in E_2 \setminus R$ coloured α. Immediately after Phase 2 is complete, we had at most $n_1 - |U|$ edges in F coloured α. Although several extra edges may have been added to F in the Presets for Phase 3, only one of them may have been coloured α. So, at the current iteration of Phase 3, we have at most $n_1 - |U| + 1 \leqslant n_2 - 2$ edges in F coloured α. As no vertex in V_2 misses α, the existence of vw is guaranteed. Hence, we remove e from F and add vw to R, uncolouring both, and add uv and $u'w$ to F, assigning to them the colour α.

In either case, the value of $d_{G_2[R]}(x) - d_{B_G[F]}(x)$ is increased only for $x = v_\beta$ and only by one. Since this value was at most $n_2 - n_1 - 2 - \mu(v_\beta)$ at the beginning of Phase 3, all $v \in V_2$ still satisfies (4.4).

It remains to prove that $G_{\overline{F,R}}$ is *Class 1*. This follows simply by observing that $V(\Lambda[G_{\overline{F,R}}]) \subseteq V_1$ (since every vertex in V_2 satisfies (4.4)), V_1 is an independent set in $G_{\overline{F,R}}$, and every graph with acyclic core is *Class 1* (Theorem 2.3 (p. 31)). $\square$

We close this section remarking how some of the results presented here are related to the results presented in Chapter 3.

Observation 4.19. *If G satisfies the preconditions of Theorem 4.18, or Theorem 4.13, or Corollary 4.14, then every major of G is strictly non-proper.*

Proof. First recall that $\Delta(G) = \Delta_1 + n_2$. Let S be the local degree sum of any major of G. It is straightforward to verify that:

Case 1. if $n_2 - n_1 \geqslant 2$ and $\Delta_1 \geqslant \Delta_2$, or if $n_1 < n_2$ and $\Delta_1 > \Delta_2$, then

$$S \leqslant (\Delta(G))^2 - 2n_2 < (\Delta(G))^2 - \Delta(G));$$

Case 2. if $m_1 \leqslant m_2 \leqslant \lceil \Delta_1(n_1 - 1)/2 \rceil$, then

$$S \leqslant (\Delta(G))^2 - (\Delta_1 + n_2^2) \leqslant (\Delta(G))^2 - \Delta(G));$$

Case 3. if $n_1 < n_2$, $\Delta_1 > \Delta_2$, and $m_1 \geqslant m_2$, then adding $n_2 - n_1$ isolated vertices to G_1 and interchanging the roles of G_1 and G_2 brings us back to Case 2. □

By the observation above, all join graphs dealt in Theorems 4.18 and 4.13 and in Corollary 4.14 are in our graph class $\mathscr{X}$, which means that these results follow immediately from Theorem 1.10. However, we have proved them in this section even so, because the proofs presented are not dependent from the recolouring procedure of Chapter 3 and yield more efficient edge-colouring algorithms than the $O(\Delta^3 nm)$-time algorithm implicit in the proof of Theorem 1.10. In fact, it can be straightforwardly verified that the worst-case time complexity of any amongst Theorems 4.18 and 4.13 and Corollary 4.14 is not asymptotically superior than $O(n_1 m_1)$ or $O(n_2 m_2)$, which are the worst-case time complexity of obtaining equitable $(\Delta_1 + 1)$-edge-colourings of G_1 and G_2, respectively (considering Vizing's usual recolouring procedure and the constructive proof of Folkman and Fulkerson (1969) for Theorem 2.8).

4.3 On a recolouring procedure for Conjecture 1.11

Let us go back to the problem of edge-colouring a join graph $G = G_1 * G_2$ with $\Delta_1 \geqslant \Delta_2$ and $\Lambda[G_1]$ acyclic. Conjecture 1.11 states that G is *Class 1*. Recall that we have proved a weaker statement in Theorem 4.9, which requires G_2 to have a sufficient number of non-majors. For the case wherein G_2 is regular, or *nearly* regular (in the sense that it does not have as many non-majors as Theorem 4.9 requires), we have achieved some further developments. These developments, which are the subject of this section, are based on a recolouring procedure similar to Vizing's recolouring procedure and to the procedure presented in Chapter 3.

Observe that, if $n_1 < n_2$, then $\Lambda[G] = \Lambda[G_1]$ and thus G is *Class 1* by Theorem 2.3. Observe also that, if $\Delta_1 > \Delta_2$, we know that G is *Class 1* (Theorem 4.7(1)). Hence, throughout this section $n_1 = n_2 =: k$ and $\Delta_1 = \Delta_2 =: d$. Recall that, if we show that G_M is *Class 1* for some perfect matching M on B_G, then G is also *Class 1* (Observation 4.8, p. 70).

In Definition 4.20, in Lemmas 4.21 and 4.23, and in Conjecture 4.22, M is a perfect matching on B_G, the edge $uv \in E(\Lambda[G_1])$, the set $\mathscr{C}$ has $d+1$ colours, and $\varphi \colon E(G_M - uv) \to \mathscr{C}$ is an edge-colouring. In these statements, the terms concerning recolouring fans and recolouring procedures are overloaded, but free of ambiguity, since we are dealing with the restricted case of the join graphs of Conjecture 1.11.

Definition 4.20. A sequence $v_0, \ldots, v_k$ of distinct neighbours of u in G_M is a *recolouring fan* for uv if $v_0 = v$ and, for all $i \in \{0, \ldots, k-1\}$: either v_i *actually* misses the colour $\alpha_i := \varphi(uv_{i+1})$; or v_i misses α_i *virtually*, that is, $i > 0$, $v_i = M(u)$, and $\varphi(wM(w)) = \alpha_i$ for some $w \in V(G_1) \setminus \{v_{i-1}\}$ which actually misses α_{i-1}. If v_k misses, actually or virtually, a colour α_k missing at u, the fan is said to be *complete*; otherwise, it is said to be *incomplete*.

Figure 4.2 on the next page illustrates a complete recolouring fan.

Lemma 4.21. *If there is a complete recolouring fan for uv, then $G_{M'}$ is* Class 1 *for some perfect matching M' on B_G.*

Proof. We shall perform the *decay of the colours*, for i from k down to 0. At the beginning of each iteration, it shall be invariant that both u and v_i miss α_i (the latter possibly

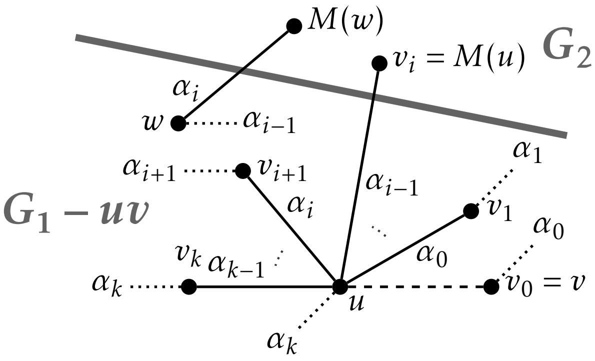


Figure 4.2: A complete recolouring fan

virtually). Note that this is true for $i = k$.

If v_i actually misses α_i, we simply assign α_i to uv_i. If $i = 0$, we are done. If $i > 0$, the vertex u now misses α_{i-1}, which is still missing (possibly virtually) at v_{i-1}, so we can continue.

If v_i misses α_i virtually, recall that $i > 0$ and $v_i = M(u)$. Then, we take $M' := (M \setminus \{uv_i, wM(w)\}) \cup \{uM(w), wv_i\}$, assigning to $uM(w)$ and uv_i the colours α_i and α_{i-1}, respectively (see Figure 4.3). Now, both u and v_{i-1} actually miss α_{i-1}. □

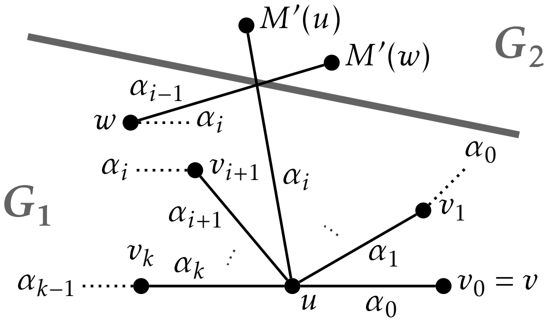


Figure 4.3: The result of the decay of the colours on the complete recolouring fan of Figure 4.2

Conjecture 4.22. *If $v_0, \ldots, v_k$ is a maximal, but not complete, recolouring fan for uv such that $v_k = M(u)$, then there are a perfect matching $\overline{M}$ on B_G and a $(d+1)$-edge-colouring of $G_{\overline{M}}$, both obtainable in polynomial time, under which there is a complete recolouring fan for uv or a maximal recolouring fan for uv starting in v_0 but not ending in $\overline{M}(u)$.*

Lemma 4.23. *If Conjecture 4.22 holds for all non-complete maximal recolouring fan for uv ending in $M(u)$, and if, for all $y \in N_{G_M}(u)$, either y misses a colour of $\mathscr{C}$, or $y = M(u)$ and $\varphi(uy)$ is missed by at least two vertices in $V(G_1)$, then $G_{M'}$ is* Class 1 *for some perfect matching M' on B_G.*

Proof. Let $F = v_0, \ldots, v_k$ be a maximal recolouring fan for uv. If F is complete, the proof follows immediately from Lemma 4.21. Otherwise, as v_0 is itself a (not necessarily complete) recolouring fan for uv, remark that $k \geqslant 0$. Moreover, if $M(u) = v_i$ for some $i \in \{1, \ldots, k\}$, the conditions of the statement imply that there must be some $w \in V(G_1) \setminus \{v_{i-1}\}$ which miss $\varphi(uv_i) = \alpha_{i-1}$. Ergo, the only reason why F is not complete is because every colour α missing (actually or virtually) at v_k is equal to α_j for some $j < k$.

Let $\alpha = \alpha_j$ for some $j < k$ be a colour missing (actually or virtually) at v_k, β be any colour missing at u, and e be the edge incident to v_k coloured β. Observe that $j < k-1$, as $\alpha_{k-1} = \varphi(uv_k)$, and also that every component of the subgraph H of G_M induced by the edges coloured α or β is a path or an even cycle. We have the following cases:

Case 1. The vertex v_k actually misses α.

Case 2. The vertex v_k misses α virtually.

In Case 1, the component of H to which e belongs is a path P, wherein v_k is one of its outer vertices. Exchanging the colours along P, we have the following subcases:

1. If the other outer vertex of P is u (which implies that $uv_{j+1} \in E(P)$), $v_j \notin V(P)$ and, thus, after the colour exchanging operation, both u and v_j miss α (the latter possibly virtually). Now, $F' := v_0, \ldots, v_j$ is a complete recolouring fan for uv, so we are done by Lemma 4.21.

2. If the other outer vertex of P is v_j, then $u \notin V(P)$ and, thus, after exchanging the colours along P, both u and v_j miss β (the latter possibly virtually). As in the previous subcase, $F' := v_0, \ldots, v_j$ is now a complete recolouring fan for uv and Lemma 4.21 applies.

3. If the other outer vertex of P is neither u nor v_j, then, after the exchanging operation, u still misses β, v_j still misses α_j, and F is thus still a recolouring fan. But now F is complete, since now v_k misses β, so we apply Lemma 4.21, but in this subcase in F instead of in F'.

In Case 2, $v_k = M(u)$ and there is some $w \in V(G_1) \setminus \{v_{k-1}\}$ which misses α_{k-1} such that $\varphi(wM(w)) = \alpha$ (see Figure 4.4). This is the case wherein our heuristic fails, but, if Conjecture 4.22 is true, we can handle this, ending up with a complete recolouring fan or going back to Case 1. ☐

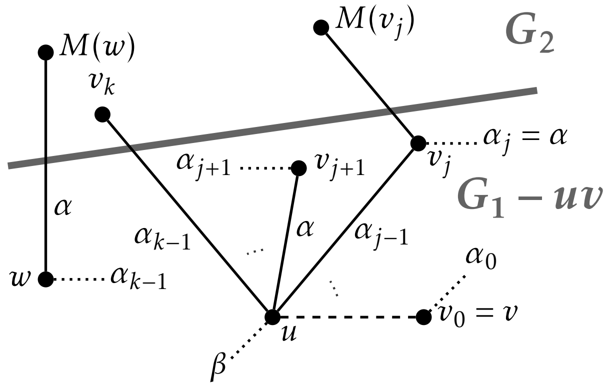


Figure 4.4: Case 2 in the proof of Lemma 4.23

Theorem 4.24. *Let G be the join of two disjoint k-order graphs G_1 and G_2 with same maximum degree d. If $\Lambda[G_1]$ is acyclic and Conjecture 4.22 is true with respect to any $uv \in E(\Lambda[G_1])$, any perfect matching M on B_G, and any non-complete maximal recolouring fan for uv ending in $M(u)$, then G is* Class 1.

Proof. We assume $d > 1$, since otherwise G_1 and G_2 are disjoint unions of cliques, in which case we already know that G is *Class 1* (Theorem 4.7(4b)). As in the proof of Theorem 4.9, for each of the components of $\Lambda[G_1]$, which are trees, choose a vertex to be the *root* of the tree and, for each $u \in V(\Lambda[G_1])$, let $h(u)$ be the *height* of u in its tree and $p(u)$ be the *parent* of u if $h(u) > 0$. Consider the non-root vertices in $V(\Lambda[G_1])$ in a non-decreasing order of height $\sigma = u_1, \ldots, u_s$. If G_2 is not regular, take a perfect matching M on B_G such that $M(u_s) \notin V(\Lambda[G_2])$. Otherwise, take any perfect matching M on B_G. Since G_M is *Class 1* if $\Lambda[G_1]$ is edgeless (Theorem 4.7(4d)), the graph $G_M - E(\Lambda[G_1])$ has an edge-colouring φ using a colour set $\mathscr{C}$ with $|\mathscr{C}| = d + 1$.

Now, take the non-root vertices in $V(\Lambda[G_1])$, one at each time, following the order σ. For each u taken, we shall colour the edge $up(u)$. This shall complete the $(d+1)$-edge-colouring of G_M, possibly replacing, at each step, the current matching in the role of M with another perfect matching on B_G. However, if G_2 is not regular, the edge $u_sM(u_s)$ shall never be replaced.

In each step of our algorithm, let u be the non-root vertex of $V(\Lambda[G_1])$ taken. The only neighbour of u which may not miss a colour of $\mathscr{C}$ is $M(u)$, because for every $x \in N_{\Lambda[G_1]}(u)$, either $x = p(u)$ or $h(x) > h(u)$, so the edge ux has not been coloured yet. If $u \neq u_s$, we have the following cases to investigate, with $\alpha := \varphi(uM(u))$:

Case 1. No vertex in G_1 misses α.

Case 2. At least two vertices in $V(G_1) \setminus \{u_s\}$ miss α.

Case 3. At most one vertex in $V(G_1) \setminus \{u_s\}$ misses α.

In Case 1, no recolouring fan for uv starting in $v_0 = p(u)$ will contain $M(u)$, which means that every vertex in the fan will miss a colour. So, we will be able to apply Vizing's usual recolouring procedure and thence colour $up(u)$.

In Case 2, since we have assumed Conjecture 4.22, we can apply Lemma 4.23 in order to colour $up(u)$, and do so preserving the edge $u_sM(u_s)$ in M.

In Case 3, we must recall that $\sum_{v\in V(G_1)}(d - d_{G_1}(v)) \geqslant d - 1$, from Lemma 4.1. As $u \neq u_s$, at least two edges of $E(\Lambda[G_1])$ have not been coloured yet, one of them being $u_sp(u_s)$. Ergo, if H is the subgraph of G_M induced only by the coloured edges, $\sum_{v\in V(G_1)\setminus\{u_s\}}((d+1) - d_H(v)) = \sum_{v\in V(G_1)}((d+1) - d_H(v)) - 1$. Furthermore,

$$\sum_{v\in V(G_1)} \Big((d+1) - d_H(v)\Big) \geqslant \sum_{v\in V(G_1)} (d - d_{G_1}(v)) + 4.$$

Consequently,

$$\sum_{v\in V(G_1)\setminus\{u_s\}} \Big((d+1) - d_H(v)\Big) \geqslant d + 2\,.$$

By the Pigeonhole Principle, this means that there must be a colour γ missed by at least two vertices in $V(G_1) \setminus \{u_s\}$. Also, it is straightforward to verify that there is some maximal path in $G_M[\alpha, \gamma]$ along whose edges the exchanging of the colours brings us back to one of the previous cases.

Finally, let us consider the last step, when $u = u_s$. Defining again H as the subgraph of G_M induced by the coloured edges, remark that

$$\sum_{v\in V(G_1)} ((d+1) - d_H(v)) \geqslant d + 1\,. \tag{4.5}$$

If G_2 is not d-regular, $M(u) \notin V(\Lambda[G_2])$ and we can apply the usual Vizing's recolouring procedure in order to colour $up(u)$. Assume then that G_2 is regular, which implies that no vertex in G_2 misses a colour of $\mathscr{C}$.

We claim that we must have at least one colour missed by at least two vertices in G_1, so the proof can follow analogously as in the previous steps. If we assume, for the sake of contradiction, that no colour of $\mathscr{C}$ is missed by more than one vertex in G_1, we have by (4.5) that every colour γ of $\mathscr{C}$ is missed by exactly one vertex v_γ. If this is true, then also $v_{\gamma_1} \neq v_{\gamma_2}$ whenever $\gamma_1 \neq \gamma_2$ for all $\gamma_1, \gamma_2 \in \mathscr{C}$, because $U \cup \{u, p(u)\}$ is a set with $d+1$ vertices of degree less than $d+1$ in H for any set $U \subset V(G_1)$ with $d-1$ vertices of degree less than d in G_1. The existence of such U is guaranteed by Lemma 4.1. But then, creating a new vertex b in H and, for all $\gamma \in \mathscr{C}$, creating the edge bv_γ in H and colouring it with γ, H would be an odd-order *Class 1* regular graph, contradicting Observation 2.17 (p. 43). Therefore, the claim holds. □

Figure 4.5 illustrates the proof of Theorem 4.24 for the join of a diamond (G_1) with a K_4 (G_2), which have both maximum degree $d = 3$. The figure depicts a perfect matching M on B_G, an edge-colouring of $G_M - E(\Lambda[G_1]) = G_M - up(u)$ with a colour set $\mathscr{C} = \{1, 2, 3, 4\}$, and a complete recolouring fan v_0, v_1, v_2 for $up(u)$.

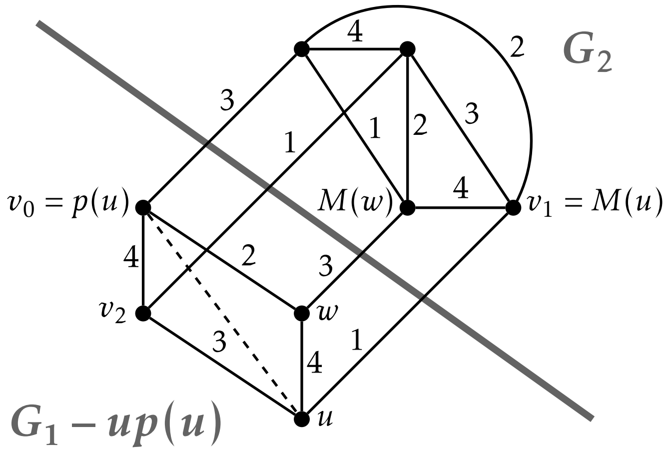


Figure 4.5: A perfect matching M on B_G when G_1 is a diamond and $G_2 = K_4$, a $(d+1)$-edge-colouring of $G_M - E(\Lambda[G_1])$, and a complete recolouring fan

4.4 On edge-colouring complementary prisms

Throughout this section, when dealing with a complementary prism $G\overline{G}$, let M be the perfect matching formed by the edges connecting the vertices of G and their corresponding vertices in $\overline{G}$.

We split the proof of Theorem 1.13 in two lemmas.

Lemma 4.25. *If G is a graph such that $\Delta(G) \neq \Delta(\overline{G})$, then $G\overline{G}$ is* Class 1.

Proof. We basically follow the same proof by De Simone and Mello (2006) for Theorem 4.7(1). We assume without loss of generality that, for our complementary prism $G\overline{G}$, we have $\Delta(G) > \Delta(\overline{G})$, so we start colouring the edges in G and in $\overline{G}$ using a colour set with cardinality $\Delta(G) + 1 = \Delta(G\overline{G})$.

We shall colours the edges of M, one at a time, to complete the proof. For each edge uv of M taken, with $u \in V(G)$, all neighbours of v are not majors of the subgraph of $G\overline{G}$ induced by the edges which have been coloured by the current moment. The proof is then concluded by applying Vizing's recolouring procedure (Lemma 1.9 (p. 21)). □

LEMMA 4.26. *If G is a non-regular graph such that $\Delta(G) = \Delta(\overline{G})$, then $G\overline{G}$ is* Class 1.

Proof. Let n and Δ be the order and the maximum degree of G, respectively, not to be confused with the order and the maximum degree of $G\overline{G}$, which are $2n$ and $\Delta + 1$, respectively. We assume $n \geqslant 4$, since it can be straightforwardly verified that the K_1 is the only graph G on at most three vertices which satisfies $\Delta(G) = \Delta(\overline{G})$.

Let also $\delta := \delta(G) = \delta(\overline{G})$. Observe that $\Delta \geqslant \delta + 1$ and all the vertices of degree δ in G correspond to the vertices of degree Δ in $\overline{G}$ and vice versa. Furthermore,

$$\Delta + \delta + 1 = n, \tag{4.6}$$

which implies

$$\Delta \geqslant \frac{n}{2} \qquad \text{and} \qquad \delta \leqslant \frac{n-2}{2}. \tag{4.7}$$

We assume for the sake of contradiction that $G\overline{G}$ is *Class 2*. So, let H be a critical subgraph of $G\overline{G}$ with the same maximum degree as $G\overline{G}$. Since $\Delta(H) = \Delta + 1$, we know that H must contain at least one edge of M incident to a major of G or to a major of $\overline{G}$. Moreover, the majors of H cannot be all in G or all in $\overline{G}$, otherwise H would be *Class 1* by taking a $\Delta(G\overline{G})$-edge-colouring of $H - (M \cap E(H))$ and then applying Vizing's recolouring at each edge of $M \cap E(H)$ in order to colour it, similarly as we have done in the proof of Lemma 4.25.

We have demonstrated that H must contain two edges uu' and vv' of M such that the vertices u and v' are majors of G and $\overline{G}$, respectively, and the vertices u' and v are vertices of degree δ in G and $\overline{G}$, respectively. Therefore, by Vizing's Adjacency Lemma (Lemma 2.4 (p. 33)), the vertex u must be adjacent in G to at least $\Delta(H) - d_H(u') + 1 \geqslant \Delta - \delta + 1$ majors of G, as the vertex v' must also be adjacent in $\overline{G}$ to at least $\Delta(H) - d_H(v) + 1 \geqslant \Delta - \delta + 1$ majors of $\overline{G}$. As u and v' are themselves majors of G and $\overline{G}$, respectively, the total number of vertices of degree Δ in G and in $\overline{G}$ must be at least $2(\Delta - \delta + 2)$, having all the other vertices of G and of $\overline{G}$ degree at least δ in G or in $\overline{G}$, respectively. Hence, in view of (4.6),

$$\begin{aligned}
|E(G\overline{G}) - M| &= \frac{1}{2}\sum_{x \in V(G)} d_G(x) + \frac{1}{2}\sum_{y \in V(\overline{G})} d_{\overline{G}}(y) \\
&\geqslant (\Delta - \delta + 2)\Delta + (n - \Delta + \delta - 2)\delta \\
&= (\Delta - \delta + 2)\Delta + (2\delta - 1)\delta \\
&= (\Delta + \delta)^2 - (\Delta - \delta + 1)\delta \\
&= (n-1)^2 - (\Delta - \delta + 1)\delta.
\end{aligned}$$

We know that $|E(G\overline{G}) - M| = n(n-1)/2$, since the identification of the corresponding vertices in G and in $\overline{G}$ forms the complete graph K_n. We claim that $(\Delta - \delta + 1)\delta < (n-1)(n-2)/2$. If this claim holds, then

$$|E(G\overline{G}) - M| = \frac{n(n-1)}{2} > (n-1)^2 - \frac{(n-1)(n-2)}{2} = \frac{n(n-1)}{2},$$

a contradiction. Therefore, the graph H cannot exist and G must indeed be *Class 1*.

Now we shall prove the claim. Observing that $(\Delta-\delta+1)\delta = -2\delta^2+n\delta$ and that, by differentiation,

$$\frac{\mathrm{d}}{\mathrm{d}\delta}(-2\delta^2+n\delta) = -4\delta+n\,,$$

we have that the function $f(\delta) := (\Delta-\delta+1)\delta$ reaches its maximum for $\delta = n/4$, which satisfies $\delta \leqslant (n-2)/2$ as required by (4.7) (recall that $n \geqslant 4$). Ergo,

$$(\Delta-\delta+1)\delta = (n-2\delta)\delta \leqslant \left(n-\frac{2n}{4}\right)\frac{n}{4} = \frac{n^2}{8}\,.$$

Since $n^2/8 < (n-1)(n-2)/2$ if and only if $3n^2-12n+8 > 0$, which is true for $n > (6+2\sqrt{3})/3$ and for $n < (6-2\sqrt{3})/3$, and since $(6+2\sqrt{3})/3 < 4$, the claim holds. □

Proof of Theorem 1.13 (p. 27). A complementary prism $G\overline{G}$ is non-regular if and only if $\Delta(G) \neq \Delta(\overline{G})$ or G is non-regular. Therefore, the proof follows immediately from Lemmas 4.25 and 4.26 and by observing that the K_2 is the *Class 1* regular complementary prism $K_1\overline{K_1}$. □

Next, we briefly comment some facts about regular complementary prisms.

Lemma 4.27. *If $G\overline{G}$ is an N-vertex D-regular complementary prism, then* $N \equiv 2 \pmod 8$, $D = (N+2)/4$, *and both G and $\overline{G}$ are d-regular graphs with* $d = (N-2)/4$ *even.*

Proof. Let n be the order of the graphs G and $\overline{G}$, not to be confused with the order of $G\overline{G}$, which is $2n =: N$. Observe that both G and $\overline{G}$ are clearly regular graphs with $d(G) = d(\overline{G}) = D-1 =: d$. Therefore, because $d(G)+d(\overline{G})+1 = n$, we have that n is odd, $d = (n-1)/2 = (N-2)/4$, and $D = (n+1)/2 = (N+2)/4$. Furthermore, since

$$\sum_{x\in V(G)} d_G(x) = \sum_{y\in V(\overline{G})} d_{\overline{G}}(y) = \frac{n(n-1)}{2}$$

must be an even number (as it is twice the number of edges in G or in $\overline{G}$), we must have $(n-1)/2$ even, which implies $n-1 \equiv 0 \pmod 4$ and $N-2 \equiv 0 \pmod 8$. □

By Lemma 4.27, the smallest regular complementary prism is the K_2 and the second smallest is the Petersen graph, which are both the only regular complementary prisms of order 2 and 10, respectively. The fact that the Petersen graph is the only 10-vertex regular complementary prism $G\overline{G}$ follows from the fact that the C_5 is the only 2-regular graph G on 5 vertices. Also by Lemma 4.27, the next regular complementary prisms are the 8 regular complementary prisms of order 18, and the next ones have order 26, and so on. There are 8 regular complementary prisms of order 18 because:

- there are 16 connected 4-regular graphs on 9 vertices (Meringer, 1997);
- there is no way that a 9-vertex 4-regular graph would be disconnected, since each component of a 4-regular graph needs at least 5 vertices;
- for each G amongst the 16 graphs on 9 vertices which are 4-regular, the graph $\overline{G}$ is also amongst these 16 graphs, so we cannot count the complementary prism $G\overline{G}$ twice.

Moreover, as it shall be clarified in the sequel, for each $N \equiv 2 \pmod 8$, we can expect exponentially many regular complementary prisms of order N, in view of the following result from the literature:

Theorem 4.28 (McKay and Wormald, 1990). *There is a constant $c > 2/3$ such that, if dn is even, if $1 \leqslant d \leqslant n-2$, and if $\min\{d, n-d\} > cn/\ln n$, then the number of d-regular graphs on n vertices is asymptotically*

$$\sqrt{2}\mathrm{e}^{\frac{1}{4}}\left(\lambda^{\lambda}(1-\lambda)^{1-\lambda}\right)^{\binom{n}{2}}\binom{n-1}{d}^{n}, \tag{4.8}$$

wherein $\lambda := d/(n-1)$.

Theorem 4.29. *For every odd $k \geqslant 3$, there is at least one k-regular complementary prism on $4k-2$ vertices, and the number of such graphs is exponential in k if k is sufficiently large.*

Proof. Lemma 4.27 implies that, for every odd k and every $(k-1)$-regular graph G on $2k-1$ vertices, the graph $G\overline{G}$ is a regular complementary prism on $4k-2$ vertices. Hence, the number of regular complementary prisms on $4k-2$ vertices is half the number of d-regular graphs on $n := 2k-1$ vertices for $d := (n-1)/2$ (recall that if G is $((n-1)/2)$-regular, then also is $\overline{G}$, yielding both graphs the same complementary prism).

Since dn is even and $1 \leqslant d \leqslant n-2$, we have by Theorem 4.28 that there is a constant $c > 2/3$ such that if $\min\{d, n-d\} = d > cn/\ln n$, then the number of d-regular graphs on n vertices is asymptotically

$$\mathrm{e}^{\frac{1}{4}}\sqrt{2}\left(\frac{1}{2}\right)^{\binom{2k-1}{2}}\binom{2k-2}{k-1}^{2k-1}. \tag{4.9}$$

Since (4.9) is exponential in k, it remains only to show that $d > cn/\ln n$, which is equivalent to show that

$$\frac{1}{2c} > \frac{n}{(n-1)\ln n}, \tag{4.10}$$

for sufficiently large n (that is, sufficiently large k, since $n = 2k-1$). But this follows immediately from the fact that $1/(2c)$ is a constant and (4.10) is decreasing (explicitly, (4.10) holds for $n > \mathrm{e}^{4c^2}$). □

Although *Class 2* regular complementary prisms are known, like the Petersen graph, we show that no complementary prism can be *SO*.

Theorem 4.30. *No complementary prism can be subgraph-overfull.*

Proof. For the sake of contradiction, assume that a complementary prism $G\overline{G}$ has an induced overfull subgraph H with $\Delta(H) = \Delta(G\overline{G})$. Moreover, H cannot be equal to $G\overline{G}$, since overfull graphs have odd order (recall Observation 2.17).

Since we have proved that the K_2 and all the non-regular complementary prisms are *Class 1*, hence not *SO*, we know that G and $\overline{G}$ are n-vertex d-regular graphs with $d = (n-1)/2$, therefore $\Delta(H) = d+1$. This implies that H must contain at least one edge of M, but not all the edges of M. Further, if $X := V(G) \cap V(H)$ and $Y := V(\overline{G}) \cap V(H)$, then

$$\partial_G(X) \cup \partial_{\overline{G}}(Y) \cup (M \cap \partial_{G\overline{G}}(V(H)) = \partial_{G\overline{G}}(V(H)).$$

Since $|\partial_{G\overline{G}}(H)| \leqslant \Delta(H)-2 = d-1$ and it cannot be the case that $\partial_G(X)$ and $\partial_{\overline{G}}(Y)$ are both empty, we must have $|\partial_G(X)| \leqslant d-2$ with $X \neq V(G)$, or $|\partial_{\overline{G}}(Y)| \leqslant d-2$ with $Y \neq V(\overline{G})$. But this is a contradiction, since it is clear that an n-vertex d-regular graph with odd n and $d \geqslant (n-1)/2$ cannot have a cut with at most $d-2$ edges. □

Theorem 4.30 brings that if G is a k-regular complementary prism, then G is an example of a non-*SO Class 2* graph. Every snark is non-*SO* and *Class 2*, but k-regular non-*SO Class 2* graphs with $k > 3$ seem not to be approached by any work in the literature. We remark that such graphs must exist, unless $\mathcal{P} = \mathcal{NP}$, since deciding if a k-regular graph is *Class 1* is an $\mathcal{NP}$-complete problem for every constant $k \geqslant 3$ (Leven and Galil, 1983).

Now, for arbitrary odd $k \geqslant 3$, consider the *Class 2* k-regular graphs wherein every cut has more than $k-2$ edges. Remark that for $k = 3$ these graphs are exactly the snarks, so this can be viewed as a natural generalisation of the definition of the snarks for arbitrary odd $k \geqslant 3$. Remark also that in the proof of Theorem 4.30 we have shown that k-regular complementary prism has odd k and every cut with more than $k-2$ edges. This motivates the *hunting* of *Class 2* regular complementary prisms. We know that not all regular complementary prisms are *Class 2*. Besides the K_2, another regular complementary prism which can be demonstrated to be *Class 1* is the graph $C_9^2\overline{C_9^2}$, wherein C_9^2 denotes the second power of the cycle C_9 (i.e. the graph defined by $V(C_9^2) := \{u_0, \ldots, u_8\}$ and $E(C_9^2) = E(C_9) \cup \{u_i u_{(i+2) \bmod 9} : 0 \leqslant i < 9\}$).

It is interesting to note, by Observation 4.5 (p. 67), that the removal of any vertex from a *Class 2* k-regular complementary prism yields another non-*SO Class 2* graph with maximum degree k.

4.5 Further decomposition techniques for edge-colouring

Theorem 1.13 in Section 4.4 can be generalised to the following result. Since the proofs for both theorems are constructive, they yield an an interesting decomposition technique for edge-colouring general graphs: if a graph G has a matching M which is also a cut in G (that is, a *separating matching*), we can decompose G into its subgraphs separated by M and construct an optimal edge-colouring of G from edge-colourings of the subgraphs, as long as M satisfies some properties.

Theorem 4.31. *Let G be a graph whose set of vertices can be partitioned into two sets A and B such that the cut between these sets is a matching M which covers all the majors of G and, for each edge $uv \in M$ with $u \in A$ and $v \in B$, the number of majors of G adjacent to u in $G[A]$ is at most $\Delta - d_G(v)$* or *the number of majors of G adjacent to v in $G[B]$ is at most $\Delta - d_G(u)$. Then G is* Class 1.

Proof. We start taking a $\Delta(G)$-edge-colouring of $G - M$, which is always possible since $\Delta(G - M) < \Delta(G) =: \Delta$. Now we shall colour the edges of M one at a time. At each edge uv considered, with $u \in A$ and $v \in B$, we assume without loss of generality that u is adjacent in $G[A]$ to at most $\Delta - d_G(v)$ majors of G. Therefore, we can colour uv by applying the claim in the proof of Vizing's Adjacency Lemma (Lemma 2.4 (p. 33)). □

Remark from the proof of Theorem 1.13 that every non-regular complementary prism satisfies the precondition of Theorem 4.31. Also, these two theorems yield the following corollaries on edge-colouring join graphs, in view of Observation 4.8.

Corollary 4.32. *Let $G_1 * G_2$ be a join graph with $n_1 \leqslant n_2$ and $\Delta_1 \geqslant \Delta_2$. If the complementary prism $G_1\overline{G_1}$ is not a regular graph distinct from the K_2, and if G_2 has a subgraph H which is isomorphic to a spanning subgraph of $\overline{G_1}$, then $G_1 * G_2$ is* Class 1.

Corollary 4.33. *Let $G_1 * G_2$ be a join graph with $n_1 \leqslant n_2$ and $\Delta_1 = \Delta_2 =: d$. If there is a maximum matching on B_G such that $|N_{G_1}(u) \cap V(\Lambda[G_1])| \leqslant d - d_{G_2}(v)$* or *$|N_{G_2}(v) \cap V(\Lambda[G_2])| \leqslant d - d_{G_1}(u)$ for each edge $uv \in M$ with $u \in V(G_1)$ and $v \in V(G_2)$, then $G_1 * G_2$ is* Class 1.

We close this chapter highlighting that graphs can be decomposed for edge-colouring also at articulation points, as clarified in Lemma 4.34, which generalises the straightforward observation that the chromatic index of any disconnected graph is the maximum amongst the chromatic indices of its connected components.

Lemma 4.34 (joint with J. P. W. Bernardi and S. M. Almeida). *The chromatic index of any graph G is the maximum amongst the degrees of the articulation points of G and the chromatic indices of its biconnected components.*

Proof. It suffices to prove that if G_1 and G_2 are any two graphs with $V(G_1) \cap V(G_2) = \{u\}$, then $\chi'(G_1 \cup G_2) = \max\{\chi'(G_1), \chi'(G_2), d_{G_1 \cup G_2}(u)\} =: k$. The proof follows by taking k-edge-colourings for G_1 and G_2 using the same set of colours $\mathscr{C}$ and then permuting $\mathscr{C}$ on G_2 so that the colours of the edges incident to u in $G_1 \cup G_2$ become all distinct. □

Remark that Lemma 4.34 implies that optimal edge-colourings can be computed in polynomial time for graphs with m edges and $O(\log m)$-size biconnected components, using for example an $O(2^m m^{O(1)})$-time exact edge-colouring algorithm at each biconnected component by Björklund et al. (2009). Lemma 4.34 also yields the result on edge-colouring chordal graphs presented in Theorem 4.36, whose proof uses the classical Menger's Theorem.

Theorem 4.35 (Menger's Theorem (Menger, 1927 *apud* Diestel, 2010)). *The size of a minimum cut set in a graph G is equal to the maximum number of disjoint paths between any pair of vertices of G.*

Theorem 4.36 (joint with J. P. W. Bernardi and S. M. Almeida). *Except for the K_3, all chordal graphs with maximum degree $\Delta \leqslant 3$ are* Class 1.

Proof. Since odd cycles are the only *Class 2* graphs with $\Delta \leqslant 2$, and since the K_3 is the only odd cycle which is chordal, from Lemma 4.34 it suffices to prove that all biconnected chordal graphs with maximum degree $\Delta \leqslant 3$ are 3-edge-colourable. In order to do so, we shall demonstrate that if G is a biconnected chordal graph with maximum degree $\Delta \leqslant 3$, then G is a subgraph of the K_4, hence 3-edge-colourable.

For the sake of contradiction, assume that G has at least five vertices. Since $\Delta \leqslant 3$ and G is biconnected, by Menger's Theorem (Theorem 4.35) there must be two non-adjacent vertices u and v in G and a cycle $C = x_0 x_1 \cdots x_t x_0$ in G for some $t \geqslant 4$ such that $u = x_0$ and $v = x_k$ for some $k \in \{2, \ldots, t-1\}$. Now, let i be the smallest integer in $\{1, \ldots, k-1\}$ such that $x_i v \in E(G)$, and let j be the greatest integer in $\{k+1, \ldots, t\}$ such that $x_j v \in E(G)$. Since G is chordal, the edges ux_i, ux_j, and $x_i x_j$ must all exist in G, which implies, since $\Delta \leqslant 3$, that C has only the four vertices u, x_i, v, x_j, which induce a diamond in G.

It is not hard to see that C is the only cycle containing u and v. If there is another cycle C', we can again demonstrate that C' has only four vertices and that these vertices induce a diamond in G. However, this would imply that $V(C) \cap V(C') = \{u, v\}$, since x_i and x_j already have degree three in C. But this would make the degrees of u and v at least four in G, a contradiction.

Since we have proved that C is the only cycle containing u and v, there must be at least one vertex x of $V(G) \setminus V(C)$ which is a neighbour of either u or v, say u, such that all paths between x and v contain u, contradicting the biconnectedness of G. $\square$

5 Some results on total colouring

In this chapter, which is organised as follows, we present the results which we have found on the subject of total colouring during the development of our work on edge-colouring:

- Section 5.1 provides further preliminaries for total colouring and other graph colouring problems which are relevant for our results;
- Section 5.2 approaches the total colouring problem when restricted to join graphs and cobipartite graphs, presenting some upper bounds for the total chromatic number of these graphs;
- Section 5.3 presents our results on total colouring circular-arc graphs, which also lead to edge-colouring results in this graph class;
- Section 5.4 closes the chapter discussing an idea about a recolouring procedure for total colouring.

5.1 Preliminaries for the chapter

A *cobipartite* graph is the complement of a bipartite graph, that is, a graph whose vertex set can be partitioned in two (disjoint and non-empty) cliques. The computational complexity of edge-colouring cobipartite graphs is an open problem, with some partial results achieved by Machado and Figueiredo (2010).

Let G be a graph and $\mathscr{C}$ be a set of t colours. A *t-edge-colouring* can be viewed as a function $\varphi\colon E(G) \to \mathscr{C}$ injective in $\partial_G(u)$ for all $u \in V(G)$. Similarly, a *t-vertex-colouring* is a function $\varphi\colon V(G) \to \mathscr{C}$ injective in $\{u, v\}$ for all $uv \in E(G)$. A *t-total colouring* is a function $\varphi\colon V(G) \cup E(G) \to \mathscr{C}$ injective in $\{u, v\}$ and injective in $\partial_G(u) \cup \{u\}$ for all $u \in V(G)$ and all $v \in N_G(u)$. The least t for which G is t-total colourable is the *total chromatic number* of G, denoted $\chi''(G)$. Obviously, $\chi''(G) \leqslant \chi(G) + \chi'(G)$.

Except for complete graphs and odd cycles, which have $\chi(G) = \Delta + 1$, the chromatic number of a graph G is at most Δ by the classical Brooks's Theorem (Brooks, 1941). Therefore, $\chi''(G) \leqslant 2\Delta + 2$. The *Total Colouring Conjecture*, proposed independently by Behzad (1965) and Vizing (1968), states that $\chi''(G) \leqslant \Delta + 2$ for every graph G. As $\chi''(G) \geqslant \Delta + 1$ by definition, graphs with $\chi''(G) = \Delta + 1$ and $\chi''(G) = \Delta + 2$ have been called *Type 1* and *Type 2*, respectively (see Figure 5.1 on the next page). The Total Colouring Conjecture was proved for some graph classes, such as complete graphs and complete bipartite graphs (Behzad et al., 1967), graphs with $\Delta \geqslant (3/4)n$ (Hilton and Hind, 1993), and dually chordal graphs (Figueiredo et al., 1999).

In particular, the complete graph K_n is *Type 1* if n is odd, or *Type 2* otherwise, and the complete bipartite graph K_{n_1,n_2} is *Type 1* if $n_1 \neq n_2$, or *Type 2* otherwise (Behzad et al., 1967). Moreover, being $V(K_n) = \{0, \ldots, n-1\}$, we call the *canonical* total colouring

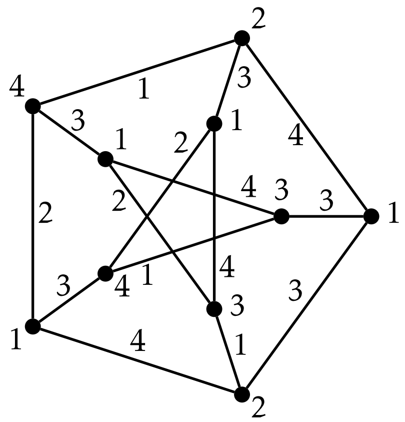


Figure 5.1: A 4-total colouring of the Petersen graph, which brings that the Petersen graph is *Type 1*.

of the K_n the function ψ_T given by

$$\psi_T(u) := (2u) \bmod (n + \text{even}(n)), \qquad \forall u \in V(K_n), \quad \text{and}$$
$$\psi_T(uv) := (u + v) \bmod (n + \text{even}(n)), \qquad \forall uv \in E(K_n),$$

wherein even(n) is 1 if n is even or 0 otherwise.

Computing the total chromatic number of a graph is an $\mathcal{NP}$-hard problem (Sánchez-Arroyo, 1989), even if restricted to bipartite graphs (McDiarmid and Sánchez-Arroyo, 1994). For a few graph classes, on the other hand, this problem can be solved in polynomial time. Examples of such graph classes are:

- complete graphs and complete bipartite graphs (Behzad et al., 1967);
- split-indifference graphs (Campos et al., 2012);
- graphs with a spanning star (Hilton, 1990);
- bipartite graphs with a spanning bistar (Hilton, 1991).

Some upper bounds for the total chromatic number of a general n-order graph G of maximum degree Δ are:

- $\chi''(G) \leqslant n + 1$ (Behzad et al., 1967);
- $\chi''(G) \leqslant \chi'(G) + 2\sqrt{\chi(G)}$ (Hind, 1990);
- $\chi''(G) \leqslant \Delta + 10^{26}$ (Molloy and Reed, 1998);
- $\chi''(G) \leqslant \Delta + 8(\ln \Delta)^8$ (Hind et al., 2000).

Now, let $\mathscr{C}$ be a set of colours, no matter how many. Under an assignment of a *list* $L(u) \subseteq \mathscr{C}$ for each $u \in V(G)$, a *vertex-list-colouring* is a vertex-colouring $\varphi\colon V(G) \to \mathscr{C}$ such that $\varphi(u) \in L(u)$ for all $u \in V(G)$. The graph G is said to be *t-vertex-choosable* if it is vertex-list-colourable under *any* assignment of lists to the vertices with at least t colours in each list. The least t for which G is t-vertex-choosable is the *vertex-choosability* of G, denoted ch(G). Analogously, under assignments of lists to the edges, we define *edge-list-colourings*, and the least t for which G is *t-edge-choosable* is the *edge-choosability* of G, denoted ch$'(G)$. Clearly, ch$(G) \geqslant \chi(G)$, ch$'(G) \geqslant \chi'(G)$, and $\chi''(G) \leqslant$ ch$'(G) + 2$.

The *Edge-List-Colouring Conjecture*[1] states that $\mathrm{ch}'(G) = \chi'(G)$ for every graph G. Remark that the similar statement concerning with vertex-list-colourings is known to be false, since one can construct a graph with $\chi(G) = 2$ and $\mathrm{ch}(G)$ arbitrarily large (Gravier, 1996), although it is true that $\mathrm{ch}(G) \leqslant \Delta + 1$ (Vizing, 1976; Erdős et al., 1979). The Edge-List-Colouring Conjecture has been shown only for a few graphs, such as the bipartite graphs (Janssen, 1993; Galvin, 1995) and the K_n with n odd (Häggkvist and Janssen, 1997) or $n-1$ prime (Schauz, 2014). For the K_n with n even and $n-1$ composite, it is only known that $\mathrm{ch}'(K_n) \leqslant \Delta(K_n) + 1 = n$ (Häggkvist and Janssen, 1997).

A *pullback* from a graph G_1 to a graph G_2 is a homomorphism $\lambda\colon V(G_1) \to V(G_2)$ injective in $\{u\} \cup N_{G_1}(u)$ for all $u \in V(G_1)$.

Theorem 5.1 (Figueiredo et al., 1997a,b, 1999). *If there is a pullback λ from G_1 to G_2, then $\chi'(G_1) \leqslant \chi'(G_2)$ and $\chi''(G_1) \leqslant \chi''(G_2)$.*

Proof. It can be verified that, if such a pullback exists and G_2 has a:

- k-edge-colouring φ, then a k-edge-colouring for G_1 can be given by

$$\psi(uv) := \varphi(\lambda(u)\lambda(v)), \qquad \forall uv \in E(G_1);$$

- k-total colouring φ, then a k-total colouring for G_1 can be given by

$$\begin{aligned} \psi(uv) &:= \varphi(\lambda(u)\lambda(v)), && \forall uv \in E(G_1), \quad \text{and} \\ \psi(u) &:= \varphi(\lambda(u)), && \forall u \in V(G_1). \end{aligned}$$

□

5.2 Upper bounds for the total chromatic number of join graphs and cobipartite graphs

This section approaches the problem of total colouring join graphs and cobipartite graphs (see Figure 5.2), for which some results have been found. These results use novel decomposition techniques and results on other graph colouring problems, mainly list-colouring problems, in order to colour each part of the decomposed graph.

Theorem 5.2. *Let G be a connected cobipartite graph with $V(G) = V_1 \cup V_2$, wherein V_1 and V_2 are two disjoint cliques with $|V_1| =: n_1$ and $|V_2| =: n_2$. Let also $B_G := G[\partial_G(V_1)] = G[\partial_G(V_2)]$, a (not necessarily complete) bipartite graph, and let $\Delta_i^B := \max_{u \in V_i} d_{B_G}(u)$ for $i \in \{1,2\}$. Then, $\chi''(G) \leqslant \max\{n_1, n_2\} + 2(\max\{\Delta_1^B, \Delta_2^B\} + 1)$.*

Proof. Let $\mathscr{C}$ be a set with $\max\{n_1, n_2\} + 2(\Delta_1^B + 1)$ colours, assuming without loss of generality that $\Delta_1^B \geqslant \Delta_2^B$. We shall construct a total colouring φ for G with $\mathscr{C}$.

Step 1. Choose n_1 colours from $\mathscr{C}$ and assign each one of them to a vertex of V_1.

Step 2. For each $uv \in E(B_G)$ with $u \in V_1$ and $v \in V_2$, create the list $L(uv)$ with any $\Delta(B_G)$ colours of $\mathscr{C}$ distinct from $\varphi(u)$. As $\Delta(B_G) = \mathrm{ch}'(B_G)$, by Galvin (1995), we can assign to each uv a colour of $L(uv)$.

[1] For more on the origin and the history of this conjecture, see Jensen and Toft (1994, Chapter 12).

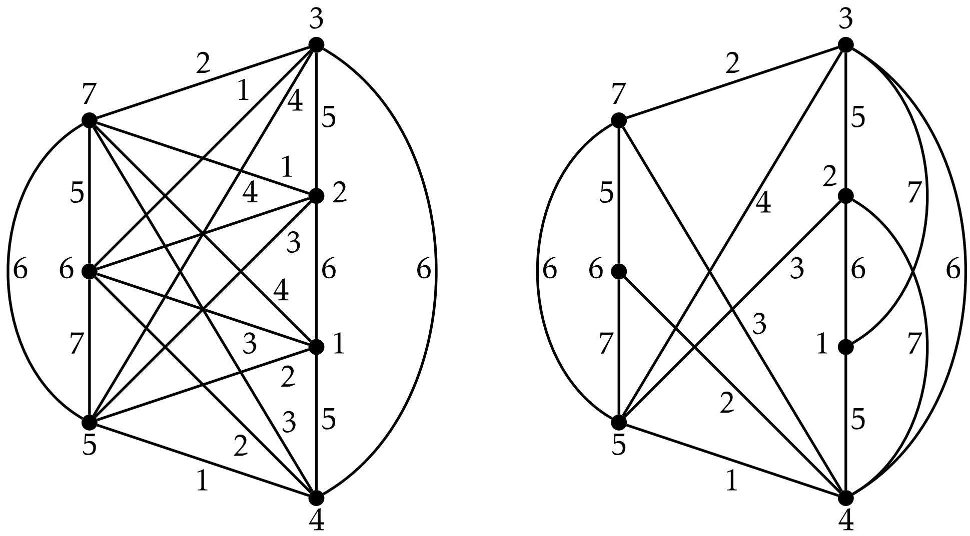


Figure 5.2: Optimal total colourings for the join graph $K_3 * C_4$ and for a cobipartite graph with $n_1 = 3$ and $n_2 = 4$. Here, differently from other figures in this text, the numbers near the vertices are the colours of those vertices.

Step 3. Now, for each $v \in V_2$, the set $X(v)$ of the colours assigned to the neighbours of v in B_G and to the edges incident to v in B_G has at most $2d_{B_G}(v)$ colours. Hence, if we take the list $L(v) := \mathscr{C} \setminus X(v)$, we have

$$|L(v)| \geqslant \max\{n_1, n_2\} + 2(\Delta_1^B + 1) - 2\Delta_2^B \geqslant n_2\,.$$

Since $\mathrm{ch}(K_{n_2}) = n_2$ is a straightforward result, we can assign to each $v \in V_2$ a colour of $L(v)$.

Step 4. Finally, in order to complete φ, it remains to colour the edges of $E(G[V_1]) \cup E(G[V_2])$. For each uv amongst them, let $X(uv)$ be the set of the colours assigned to the vertices u and v and to the edges of B_G adjacent to uv in G. Define then the list $L(uv) := \mathscr{C} \setminus X(uv)$. Since $|X(uv)| \leqslant 2\Delta_1^B + 2$, $|L(uv)| \geqslant \max\{n_1, n_2\}$. Thus, by the result of Häggkvist and Janssen (1997) according to which $\mathrm{ch}'(K_n) \leqslant n$, we can assign to each $uv \in E(G[V_1]) \cup E(G[V_2])$ a colour of $L(uv)$. □

Because all the colourings in the proof of Theorem 5.2 can be obtained in polynomial time, our proof yields a polynomial-time algorithm to construct a $(\max\{n_1, n_2\} + 2(\max\{\Delta_1^B, \Delta_2^B\} + 1))$-total colouring. Recall that $\Delta(G) = \max\{n_1 - 1 + \Delta_1^B, n_2 - 1 + \Delta_2^B\}$, which means that the upper bound provided in Theorem 5.2 is better than the bounds for general graphs by Behzad et al. (1967), Hind (1990), Molloy and Reed (1998), and Hind et al. (2000), as long as Δ_1^B and Δ_2^B are not *too large*, in the sense that the observation below clarify.

Observation 5.3. *The upper bound* $(\max\{n_1, n_2\} + 2(\max\{\Delta_1^B, \Delta_2^B\} + 1))$ *for* $\chi''(G)$ *provided by Theorem 5.2 is strictly less than:*

(i) $|V(G)| + 1$, *the upper bound for* $\chi''(G)$ *by Behzad et al. (1967), as long as*

$$\max\{\Delta_1^B, \Delta_2^B\} \leqslant \frac{\min\{n_1, n_2\}}{2} - 1\,;$$

(ii) $\Delta(G) + 10^{26}$, *the upper bound for* $\chi''(G)$ *by Molloy and Reed (1998), as long as*

$$\max\{\Delta_1^B, \Delta_2^B\} \leqslant 5 \times 10^{25} - 2\,;$$

(iii) $\chi'(G)+2\sqrt{\chi(G)}$, *the upper bound for* $\chi''(G)$ *by Hind (1990), as long as*

$$\max\{\Delta_1^B,\Delta_2^B\} \leqslant \sqrt{\max\{n_1,n_2\}}-\frac{3}{2};$$

(iv) $\Delta(G)+8(\ln\Delta(G))^8$, *the upper bound for* $\chi''(G)$ *by Hind et al. (2000), as long as*

$$\max\{\Delta_1^B,\Delta_2^B\} \leqslant 4\Big(\ln(\max\{n_1+\Delta_1^B,n_2+\Delta_2^B\})\Big)^8-\frac{3}{2}.$$

Proof. Follows by observing that:

(i) if $\max\{\Delta_1^B,\Delta_2^B\} \leqslant (\min\{n_1,n_2\})/2-1$, then

$$\begin{aligned}\max\{n_1,n_2\}+2(\max\{\Delta_1^B,\Delta_2^B\}+1) &\leqslant \max\{n_1,n_2\}+\min\{n_1,n_2\}\\ &= n_1+n_2<|V(G)|+1\,;\end{aligned}$$

(ii) if $\max\{\Delta_1^B,\Delta_2^B\} \leqslant 5\times 10^{25}-2$, then

$$\begin{aligned}\max\{n_1,n_2\}+2(\max\{\Delta_1^B,\Delta_2^B\}+1) &\leqslant \max\{n_1-1+\Delta_1^B,n_2-1+\Delta_2^B\}+10^{26}-2\\ &< \Delta(G)+10^{26}\,;\end{aligned}$$

(iii) if $\max\{\Delta_1^B,\Delta_2^B\} \leqslant \sqrt{\max\{n_1,n_2\}}-3/2$, then

$$\begin{aligned}\max\{n_1,n_2\}+2(\max\{\Delta_1^B,\Delta_2^B\}+1) &\leqslant \Delta(G)+2\sqrt{\max\{n_1,n_2\}}-1\\ &< \Delta(G)+2\sqrt{\chi(G)} \leqslant \chi'(G)+2\sqrt{\chi(G)}\,;\end{aligned}$$

(iv) if $\max\{\Delta_1^B,\Delta_2^B\} \leqslant 4\Big(\ln(\max\{n_1+\Delta_1^B,n_2+\Delta_2^B\})\Big)^8-3/2$, then

$$\max\{n_1,n_2\}+2(\max\{\Delta_1^B,\Delta_2^B\}+1) \leqslant \Delta(G)+8(\ln(\Delta(G)))^8-1\,. \qquad \square$$

In Theorem 5.4 below, we use $\square_i$, for $\square \in \{\chi,\chi',\chi''\}$ and $i\in\{1,2\}$, to denote $\square(G_i)$, for the sake of simplicity,

Theorem 5.4. *Let* G *be the join of two disjoint graphs* G_1 *and* G_2 *with, respectively,* n_1 *and* n_2 *vertices and maximum degrees* Δ_1 *and* Δ_2*, but without assuming this time that* $n_1 \leqslant n_2$*. Let* B_G *be the complete bipartite graph* $G-(E_1\cup E_2)$*. Let also*

$$P(G_1,G_2) := \min\{\Delta_1+\Delta_2+1,\max\{\chi_1',\chi_2''\}\}. \tag{5.1}$$

Then, $\chi''(G) \leqslant \max\{n_1,n_2\}+1+P(G_1,G_2)$*.*

Proof. Recall that $\Delta(G)=\max\{\Delta_1+n_2,\Delta_2+n_1\}$. Let $t:=\max\{n_1,n_2\}+1+P(G_1,G_2)$ and take two disjoint sets $\mathscr{C}_A$ and $\mathscr{C}_B$ with, respectively, χ_1 and $\max\{\chi_1',\chi_2''\}$ colours. As it can be straightforwardly verified that $|\mathscr{C}_A|+|\mathscr{C}_B|\leqslant t$, take a set $\mathscr{C}$ with t colours having $\mathscr{C}_A$ and $\mathscr{C}_B$ as subsets. We shall construct a total colouring $\varphi\colon V(G)\cup E(G)\to\mathscr{C}$.

Step 1. Take a χ_1-vertex-colouring of G_1 using only the colours of $\mathscr{C}_A$.

Step 2. Take a χ'_1-edge-colouring of G_1 and a χ''_2-total colouring of G_2, both using only the colours of $\mathscr{C}_B$. Since $\mathscr{C}_A$ and $\mathscr{C}_B$ are disjoint, no colour conflict has been created.

Step 3. Now, for each edge $uv \in B_G$, with $u \in V_1$, let $X(uv)$ be the set of the colours assigned to the vertices u and v and to the edges of $G_1 \cup G_2$ adjacent to uv in G. It is clear that $|X(uv)| \leqslant 1 + P(G_1, G_2)$. Define then the list $L(uv) := \mathscr{C} \setminus X(uv)$. Since $|L(uv)| \geqslant t - 1 - P(G_1, G_2) = \max\{n_1, n_2\}$ and $\mathrm{ch}'(B_G) = \max\{n_1, n_2\}$ (Galvin, 1995), we can assign to each $uv \in E(B_G)$ a colour of $L(uv)$. □

Remark in Theorem 5.4 that, from the definition of $P(G_1, G_2)$ in (5.1), the choice of the graphs for the roles of G_1 or G_2 may lead to different upper bounds. Moreover, if $P(G_1, G_2)$ is known, or if it can be computed in polynomial time, then our proof is a polynomial-time algorithm, provided that the underlying colourings are also known or can be computed. Replacing $P(G_1, G_2)$ by some upper bound on it, such as $\Delta_1 + \Delta_2 + 1$, also makes our algorithm polynomial.

Similar to the bound for the cobipartite graphs, the upper bound presented in Theorem 5.4 is better than the upper bounds for general graphs if $P(G_1, G_2)$ is not *too large*, in the sense that the observation below clarifies.

Observation 5.5. *The upper bound* $\max\{n_1, n_2\} + 1 + P(G_1, G_2)$ *for* $\chi''(G)$ *provided by Theorem 5.2 is strictly less than:*

(i) $|V(G)| + 1$, *the upper bound for* $\chi''(G)$ *by Behzad et al. (1967), as long as*

$$P(G_1, G_2) \leqslant \min\{n_1, n_2\} - 1\,;$$

(ii) $\Delta(G) + 10^{26}$, *the upper bound for* $\chi''(G)$ *by Molloy and Reed (1998), as long as*

$$P(G_1, G_2) \leqslant 10^{26} - 1\,;$$

(iii) $\chi'(G) + 2\sqrt{\chi(G)}$, *the upper bound for* $\chi''(G)$ *by Hind (1990), as long as*

$$P(G_1, G_2) \leqslant 2\sqrt{\chi_1 + \chi_2} - 1\,;$$

(iv) $\Delta(G) + 8(\ln \Delta(G))^8$, *the upper bound for* $\chi''(G)$ *by Hind et al. (2000), as long as*

$$P(G_1, G_2) \leqslant 8\Big(\ln(\max\{n_1 + \Delta_1^B, n_2 + \Delta_2^B\})\Big)^8 - 1\,.$$

Proof. Follows by simply observing that:

(i) if $P(G_1, G_2) \leqslant \min\{n_1, n_2\} - 1$, then

$$\max\{n_1, n_2\} + 1 + P(G_1, G_2) \leqslant \max\{n_1, n_2\} + \min\{n_1, n_2\} = |V(G)|;$$

(ii) if $P(G_1, G_2) \leqslant 10^{26} - 1$, then

$$\max\{n_1, n_2\} + 1 + P(G_1, G_2) < \max\{n_1 + \Delta_2, n_2 + \Delta_1\} + 10^{26}\,;$$

(iii) if $P(G_1,G_2) \leqslant 2\sqrt{\chi_1+\chi_2}-1$, then

$$\begin{aligned}\max\{n_1,n_2\}+1+P(G_1,G_2) &< \max\{n_1+\Delta_2,n_2+\Delta_1\}+2\sqrt{\chi_1+\chi_2}\\ &\leqslant \chi'(G)+2\sqrt{\chi(G)};\end{aligned}$$

(iv) if $P(G_1,G_2) \leqslant 8\Big(\ln(\max\{n_1+\Delta_1^B,n_2+\Delta_2^B\})\Big)^8-1$, then

$$\max\{n_1,n_2\}+1+P(G_1,G_2) < \max\{n_1+\Delta_2,n_2+\Delta_1\}+8(\ln\Delta(G))^8. \qquad \square$$

From now on in this section, $G = G_1 * G_2$ is a join graph. Inspired by the observation by De Simone and Mello (2006) according to which G is *Class 1* whenever G_M is *Class 1* for some maximal matching M on B_G, we show how the upper bound of Theorem 5.4 may be lowered in some cases. In the statements, as it us usual for functions $f : A \to B$ and $X \subseteq A$, we use $f(X)$ to denote the set $\bigcup_{x\in X} f(x)$.

Theorem 5.6. *Let φ be a total colouring of G_M for some perfect matching M on B_G. If the sets $\varphi(V_1)$ and $\varphi(E_1\cup M\cup V_2\cup E_2)$ are disjoint and*

$$|\varphi(E_1\cup M\cup V_2\cup E_2)| \leqslant \max\{\chi_1',\chi_2''\} \leqslant \Delta_1+\Delta_2+3,$$

then $\chi''(G) \leqslant \max\{n_1,n_2\}+\max\{\chi_1',\chi_2''\}$.

Proof. Let $\mathscr{C}$ be a set with $t := \max\{n_1,n_2\}+\max\{\chi_1',\chi_2''\}$ colours having $\mathscr{C}_A := \varphi(V_1)$ and $\mathscr{C}_B := \varphi(E_1\cup M\cup V_2\cup E_2)$ as subsets. In order to obtain a t-total colouring of G using the colours of $\mathscr{C}$, we start with the total colouring φ of G_M, remaining to colour only the edges of $B_G - M$.

We proceed now as in Step 3 of the proof of Theorem 5.4. For each edge $uv \in B_G - M$, with $u \in V_1$, let $X(uv)$ be the set of the colours assigned to the vertices u and v and to the edges of G_M adjacent to uv in G. Clearly

$$\begin{aligned}|X(uv)| &\leqslant 1+\min\{\Delta_1+\Delta_2+3,\max\{\chi_1',\chi_2''\}\}\\ &= 1+\max\{\chi_1',\chi_2''\}.\end{aligned}$$

Therefore, if we define the list $L(uv) := \mathscr{C}\setminus X(uv)$, we have $|L(uv)| = \max\{n_1,n_2\}-1 = \Delta(B_G)$. Since $\Delta(B_G) = \mathrm{ch}'(B_G)$ Galvin (1995), we can assign to each $uv \in E(B_G)$ a colour of $L(uv)$. $\square$

Corollary 5.7. *If G has a total colouring φ of G_M, for some perfect matching M on B_G, satisfying the preconditions of Theorem 5.6, and if* $\max\{n_1,n_2\}+\max\{\chi_1',\chi_2''\} \leqslant \max\{n_1+\Delta_2+2,n_2+\Delta_1+2\}$, *then the Total Colouring Conjecture holds for G, i.e. $\chi''(G) \leqslant \Delta(G)+2$.*

Theorem 5.8 below uses pullback functions in order to lower the bound for the total chromatic number of join graphs.

Theorem 5.8. *If there is a graph G_3 such that*

1. $\max\{\chi_3'',\Delta_3+2\} \leqslant \Delta_1+\Delta_2+3$,

2. *there are two pullbacks λ_{13} and λ_{23} to G_3, the former from G_1 and the latter from G_2, and*

3. *there is a perfect matching M on B_G such that $\lambda_{13}(u) = \lambda_{23}(v)$ for all $uv \in M$ with $u \in V_1$,*

then $\chi''(G) \leqslant \max\{n_1, n_2\} + \max\{\chi_3'', \Delta_3 + 2\}$.

Proof. Let $\mathscr{C}_A$ be a set with χ_1 colours and take any optimal vertex-colouring of G_1. Let $\mathscr{C}_B$ be a set with $\max\{\chi_3'', \Delta_3 + 2\}$ colours, disjoint from $\mathscr{C}_A$, and ψ be a total colouring of G_3 using the colours of $\mathscr{C}_B$. By Theorem 5.8, the function $\varphi_1 \colon E_1 \to \mathscr{C}_B$ defined by

$$\varphi_1(uv) = \psi(\lambda_{13}(u)\lambda_{13}(v)), \qquad \forall uv \in E_1,$$

is a proper edge-colouring of G_1, as the function $\varphi_2 \colon V_2 \cup E_2 \to \mathscr{C}_B$ defined by

$$\begin{aligned} \varphi_2(u) &:= \psi(\lambda_{23}(u)), & \forall u \in V_2, \\ \varphi_2(uv) &:= \psi(\lambda_{23}(u)\lambda_{23}(v)), & \forall uv \in E_2, \end{aligned}$$

is a proper total colouring of G_2. Since it is clear that $\max\{\chi_3'', \Delta_3+2\} > \Delta_3+1$, at least one colour $\alpha_x \in \mathscr{C}_B$ is *missing at* each $x \in V_3$, i.e. α_x is not the colour assigned by ψ to x nor to any edge incident to x. Ergo, for all $uv \in M$ with $u \in V_1$, the colour $\alpha_{f(u)}$ is missing at both u and v and thence can be assigned to uv. This yields a $(\max\{\chi_3'', \Delta_3 + 2\})$-total colouring φ of G_M with $\varphi(V_1)$ and $\varphi(E_1 \cup M \cup V_2 \cup E_2)$ disjoint and

$$\begin{aligned} |\varphi(E_1 \cup M \cup V_2 \cup E_2)| &\leqslant \max\{\chi_3'', \Delta_3 + 2\} \\ &\leqslant \Delta_1 + \Delta_2 + 3\,. \end{aligned}$$

The rest of the proof follows as the proof for Theorem 5.6, but using $t = \max\{n_1, n_2\} + \max\{\chi_3'', \Delta_3 + 2\}$ instead of $t = \max\{n_1, n_2\} + \max\{\chi_1', \chi_2''\}$. □

Corollary 5.9. *If there is a graph G_3 satisfying the preconditions of Theorem 5.8, and if $\max\{n_1, n_2\} + \max\{\chi_3'', \Delta_3 + 2\} \leqslant \max\{n_1 + \Delta_2, n_2 + \Delta_1\} + 2$, then the Total Colouring Conjecture holds for G.*

Theorem 5.10 and Corollary 5.11 deal with the joins of indifference graphs.

Theorem 5.10. *If G_1 and G_2 are indifference graphs, then*

$$\chi''(G) \leqslant \max\{n_1, n_2\} + \max\{\Delta_1, \Delta_2\} + 2\,.$$

Proof. The proof follows from Theorem 5.8 by taking $G_3 := K_{\max\{\Delta_1, \Delta_2\}+1}$. By Figueiredo et al. (1997b), if $u_0, \ldots, u_{k-1}$ is an indifference order of an indifference graph, then, for any $\ell > k$, the function given by $\lambda(u_i) := i \bmod \ell$ is a pullback from this indifference graph to the K_ℓ on vertex set $\{0, \ldots, \ell - 1\}$. Therefore, back to our join graph G, it is clear that a matching M on B_G satisfying the requirements of Theorem 5.8 can be taken. □

Corollary 5.11. *If G_1 and G_2 are indifference graphs, and if $n_1 = n_2$ or $\Delta_1 = \Delta_2$, then the Total Colouring Conjecture holds for G.*

5.3 On total and edge-colouring proper circular-arc graphs

Throughout this section, let G be an n-vertex proper circular-arc graph with maximum degree Δ, being $\sigma := u_0, \ldots, u_{n-1}$ a proper circular-arc order of G and, $0, \ldots, \ell-1$ the vertices of the K_ℓ, for every positive integer ℓ. Remark that when we say that G is $(\Delta + 2)$-total colourable, it does not mean that G is *Type 2*.

Theorem 5.12 (joint with J. P. W. Bernardi and S. M. Almeida). *If* $n \equiv 0 \pmod{(\Delta+1)}$, *then* G *is:* Class 1 *and* $(\Delta+2)$*-total colourable if* Δ *is odd;* Type 1 *if* Δ *is even.*

Proof. It suffices to show that if $n \equiv 0 \pmod{(\Delta+1)}$, then there is a pullback from G to the $K_{\Delta+1}$. Assume, for the sake of contradiction, that the function $\lambda\colon V(G) \to V(K_{\Delta+1})$ defined by $\lambda(u_i) := i \bmod (\Delta+1)$ is *not* a pullback from G to the $K_{\Delta+1}$. As λ is clearly a homomorphism, there must be two distinct vertices v_1 and v_2 in $V(G)$ which have a neighbour w in common and satisfy $\lambda(v_1) = \lambda(v_2)$. However, since σ is a proper circular-arc order of G, all vertices between v_1 and v_2 in σ are thus neighbours of w, which straightforwardly implies $d_G(w) > \Delta$. □

Theorem 5.13 (joint with J. P. W. Bernardi and S. M. Almeida). *If* $n \not\equiv k \pmod{(\Delta+1)}$, *for all* $k \in \{1, \Delta\}$, *and if* G *has a maximal clique of size two, then* G *is:* Class 1 *and* $(\Delta+2)$*-total colourable if* Δ *is odd;* Type 1 *if* Δ *is even.*

Proof. We already know that the theorem holds if G is an indifference graph (Figueiredo et al., 1997b, 1999). So we assume that G is not an indifference graph, which implies that its maximum clique of size two is not a bridge.

If $r := n \bmod (\Delta+1) = 0$, we are done by Theorem 5.12. If $\Delta \leqslant 2$, then G is a cycle or a disjoint union of paths and the theorem clearly holds. Hence, we assume $\Delta \geqslant 3$ and $r \neq 0$. We also assume without loss of generality that $\{u_0, u_{n-1}\}$ is a maximal clique. Because σ is a proper circular-arc order, we have $u_\Delta \notin N_G(u_0)$ and $u_{n-1-\Delta} \notin N_G(u_{n-1})$, otherwise $d_G(u_0) > \Delta$ or $d_G(u_{n-1}) > \Delta$.

Let $\varphi \in \{\psi_2, \psi_T\}$ be the canonical total or edge-colouring of the $K_{\Delta+1}$. Let also $G' := G - u_{n-1}u_0$, which is clearly a connected indifference graph. Then, the function $\lambda\colon V(G') \to V(K_{\Delta+1})$ defined by $\lambda(u_i) := i \bmod (\Delta+1)$ is clearly a pullback from G' to the $K_{\Delta+1}$ and brings a total or an edge-colouring ψ of G' using the same set of colours as φ. Ergo, we have only to colour $u_{n-1}u_0$ in order to complete the proof.

Observe that $\lambda(u_{n-1}) = r-1$, $\lambda(u_{n-1-\Delta}) = r$, and, since neither r nor $r-1$ is Δ,

$$\varphi(r, r-1) = \begin{cases} (2r-1) \bmod \Delta, & \text{if } \varphi = \psi_2; \\ (2r-1) \bmod (\Delta + 1 + \operatorname{even}(\Delta+1)), & \text{if } \varphi = \psi_T. \end{cases}$$

Let $q := \varphi(r, r-1)$. As $\lambda(v) \neq \Delta$ and $\lambda(w) \neq r$ for all $v \in N_{G'}(u_0)$ and all $w \in N_{G'}(u_{n-1})$, the colour $\varphi(0,\Delta)$ is missing at u_0 and the colour q at u_{n-1}. If $q = \varphi(0,\Delta)$, then we assign the colour q to $u_{n-1}u_0$ and we are done. Otherwise, since $q \in \{0, \ldots, \Delta\}$, we exchange Δ and q in the codomain of λ, that is, we redefine λ so that every vertex which has been mapped by λ to Δ is now mapped to q and vice versa. Notice that the images of u_0, $u_{n-1-\Delta}$, and u_{n-1} by λ remain the same, but $\lambda(u_\Delta)$ becomes q, which now is also a colour missing at u_0. Then, we colour $u_{n-1}u_0$ with q. □

Let $\mathscr{A}$ be the class of the proper circular-arc graphs with odd Δ and a maximal clique of size two. Overfull graphs in $\mathscr{A}$ can be constructed when $n \equiv 1 \pmod{(\Delta+1)}$ and when $n \equiv \Delta \pmod{(\Delta+1)}$ (see Figures 5.3(a) and 5.3(b), respectively). From Theorem 5.13 follows an interesting corollary on the *SO* graphs in $\mathscr{A}$:

Corollary 5.14 (joint with J. P. W. Bernardi and S. M. Almeida). *A graph in* $\mathscr{A}$ *is* SO *if and only if it is overfull.*

Proof. Let G be a graph in $\mathscr{A}$. If G is overfull, then G is *SO* by definition. So, we assume that G has an induced overfull Δ-subgraph H and, for the sake of contradiction, that

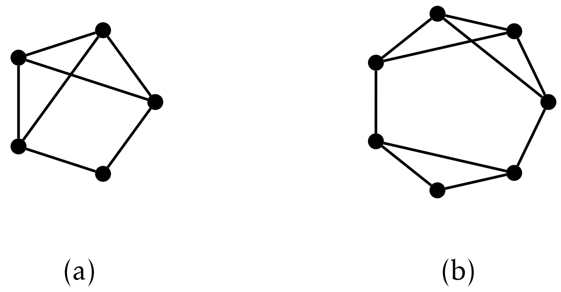


Figure 5.3: Two overfull graphs in $\mathscr{A}$

$V(H) \neq V(G)$. We also assume $\Delta > 2$, since the only *SO* graphs in $\mathscr{A}$ with $\Delta \leqslant 2$ are the odd cycles, which are overfull.

Remark that every subgraph of G which has H as a subgraph is also *SO*. Hence, for every $x \in V(G) \setminus V(H)$, the subgraph of G induced by $V(H) \cup \{x\}$ must be *Class 2*. However, Theorem 5.13 brings that either $|V(H)| \equiv 1 \pmod{(\Delta + 1)}$ or $|V(H)| \equiv \Delta \pmod{(\Delta + 1)}$. Therefore, since $\Delta > 2$, we have $|V(H) \cup \{x\}| \not\equiv k \pmod{(\Delta + 1)}$ for all $k \in \{1, \Delta\}$, which implies, also by Theorem 5.13, that G is *Class 1*, a contradiction. □

5.4 On a recolouring procedure for total colouring

A standard result on vertex-colouring is that any greedy (not necessarily optimal) vertex-colouring algorithm does not need more than $\Delta + 1$ colours to colour the vertices of a graph G with maximum degree Δ. As we have already mentioned, only complete graphs and odd cycles have $\chi(G) = \Delta + 1$. All the other graphs satisfy $\chi(G) \leqslant \Delta + 1$ by Brooks's Theorem (Brooks, 1941). In this section we discuss a polynomial-time *heuristic* for constructing edge by edge a $(\Delta + k)$-total colouring of G over an initial $(\Delta + k)$-vertex-colouring, for any $k \geqslant 2$. Our heuristic (which may fail, as clarified in the sequel) is an attempt based on a recolouring procedure similar to Vizing's recolouring procedure for edge-colouring.

As an edge-colouring can be regarded as a vertex-colouring of a line graph, a total colouring can be regarded as a vertex-colouring of a *total graph*. The *total graph* of a graph G (see Figure 5.4 on the next page), denoted $\mathrm{T}(G)$, is the graph defined by

$$V(\mathrm{T}(G)) := V(G) \cup E(G) \quad \text{and}$$
$$E(\mathrm{T}(G)) := E(G) \cup E(\mathrm{L}(G)) \cup \{ue : u \in V(G) \text{ and } e \in \partial_G(u)\}.$$

In the context of total colourings, it is quite usual in the literature to say that a vertex $v \in V(G)$ *misses* some $\alpha \in \mathscr{C}$ in a total colouring $\varphi\colon V(G) \cup E(G) \to \mathscr{C}$ if no element of $\{v\} \cup \partial_G(v)$ is coloured α. Remark that saying that α is missing at v does not necessarily mean that no neighbour of v is coloured α. Furthermore, being α a colour *not* missing at some $u \in V(G)$ and β a colour *not* missing at some $v \in V(G)$, we say that u and v are *α/β-connected* if *the* α-coloured element of $\{u\} \cup \partial_G(u)$ and *the* β-coloured element of $\{v\} \cup \partial_G(v)$ are in the same component (a bipartite graph B) of the subgraph of $\mathrm{T}(G)$ induced by the elements coloured α or β.

In this section and only for it, we redefine the concept of a *recolouring fan*, now for the context of total colouring, with no risk of ambiguity.

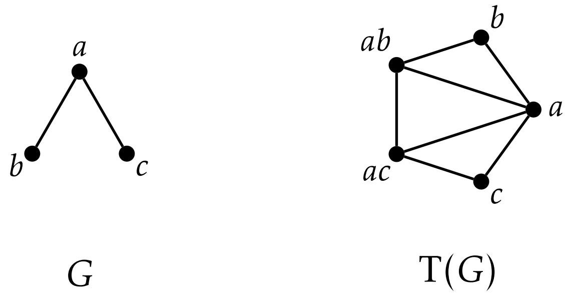


Figure 5.4: A graph and its total graph

Definition 5.15. Let uv be an edge of a graph G and let $\varphi\colon (V(G)\cup E(G))\setminus\{uv\}\to\mathscr{C}$ be a total colouring of $G-uv$. A *recolouring fan* for uv is a sequence $v_0,\ldots,v_k$ of distinct neighbours of u such that $v_0=v$ and, for all $i\in\{0,\ldots,k-1\}$, the colour $\alpha_i := \varphi(uv_{i+1})$ is *missing* at v_i. Moreover, the recolouring fan is said to be *complete* if

(i) either there is some $\beta\in\mathscr{C}$ missing at both u and v_k;

(ii) or for some $\alpha\in\mathscr{C}$ missing at v_k and some $\beta\in\mathscr{C}$ missing at u, the vertices u and v_k are not α/β-connected.

Theorem 5.16. *Let uv be an edge of a graph G and let $\varphi\colon (V(G)\cup E(G))\setminus\{uv\}\to\mathscr{C}$ be a $(\Delta+k)$-total colouring of $G-uv$ for some positive integer k. If there is a complete recolouring $v_0,\ldots,v_k$ for uv, then G is also $(\Delta+k)$-total colourable.*

Proof. If (i) there is some $\beta\in\mathscr{C}$ missing at both u and v_k, then we can perform the decay of the colours of the edges of the recolouring fan as in the proof for Lemma 1.6 (p. 20). On the other hand, if (i) does not hold, but (ii) for some $\alpha\in\mathscr{C}$ missing at v_k and some $\beta\in\mathscr{C}$ missing at u, the vertices u and v_k are not α/β-connected, then let H be the subgraph of $\mathrm{T}(G)$ induced by the elements coloured α or β, and let B (a bipartite graph) be the component of H cont containing v_k. Exchanging the colours of the elements of B brings us back to (i). □

In view of Theorem 5.16, our $(\Delta+k)$-total colouring heuristic consists simply of greedily colouring the vertices of the graph G with $\Delta+k$ colours and then trying to colour each edge uv of G, one at a time, by constructing a complete recolouring fan for uv. Of course, our heuristic may fail, even given that $k\geqslant 2$, which implies that there shall always be at least one colour missing at each vertex. The reason why a similar approach works for edge-colourings, from which Vizing successfully derived his theorem (Theorem 1.4, p. 19), is that, when we take two colours α and β and look at the edges coloured α or β, they form paths or even cycles, as it has been widely explored in this text. From this fact, we can affirm, like in the proof of Lemma 1.8 (p. 20), that only one amongst two vertices v_j and v_k can be in the same α_j/β-component as u, for some β missing at u and some α_j missing at both v_j and v_k. In the other hand, in an analogous situation in the context of total colourings, it may be the case wherein the vertices u, v_j, and v_k are all α_j/β-connected.

Although the heuristic presented may fail, some important questions concerning it arise, for which we encourage future investigation. We discuss some of these questions in the final remarks of this document (Chapter 6).

6 Conclusion

Back to Alice's problem described in the introductory chapter, our fictional character now knows that, although her problem is *hard* (that is, *$\mathcal{NP}$-hard*) for graphs in general, it may be the case wherein her graph happens to belong to a class for which the problem is already known to be *easy* (that is, *polynomial*). In this chapter, we summarise the state of the art of her problem then and now, after the novel results presented in this thesis, restricted to the graph classes which we have approached. Also, we further discuss the conjectures and open problems left for future investigation.

6.1 Final remarks on edge-colouring graphs with bounded local degree sums

In Chapter 1, we have introduce the graph class $\mathscr{X}$, which is the class of the graphs whose majors have local degree sum at most $\Delta^2 - \Delta$. We have proved in Chapter 3 that all graphs in $\mathscr{X}$ are *Class 1* and that almost every graph is in $\mathscr{X}$. Moreover, we have presented a novel recolouring procedure to construct in $O(\Delta^3 nm)$-time a Δ-edge-colouring of any graph in $\mathscr{X}$. In our proofs, we have concerned ourselves only with constructiveness and polynomiality, thus choosing not to complicate the presentation of the proofs just to achieve some better time complexity. It would be interesting to investigate data structures and strategies that may lead to a more efficient algorithm.

We suspect the following slightly stronger form of Theorem 1.10.

Conjecture 6.1. *Every graph with no proper majors is* Class 1.

Recall that saying that a graph G has no proper major is the same as saying that the majors of G have local degree sum bounded above by $\Delta^2 - \Delta + 1$. Therefore, the graph class $\mathscr{X}$ is a subclass of the class of the graphs with no proper majors. Recall also that we have proved that Conjecture 6.1 holds for triangle-free graphs (Theorem 3.11).

Table 6.1 gathers some graph classes defined by upper bounds on the local degree sums of the major vertices. In all the tables presented in this chapter, we discriminate for which graph (sub)classes the problem has been solved (that is, in our context, a polynomial-time edge-colouring algorithm has been presented) and for which classes the computational complexity of the problem remains open.

Table 6.1: State of the art then and now on edge-colouring graphs with bounded local degree sums. The ∅ symbol indicates that the corresponding graph classes had not been approached by any work in the literature, to the best of our knowledge.

	Then		Now	
Graph class	Status	References	Status	Theorems
graphs with no proper majors	open	∅	open	
↳ graphs in $\mathscr{X}$	open	∅	solved	1.10 (p. 24)
↳ which are triangle-free	open	∅	solved	3.11 (p. 58)

We have come to the study of graphs with no proper majors and to the development of the extended recolouring procedure presented in Chapter 3 mainly from our efforts to approach Conjecture 6.1. Proving this conjecture may be an important step towards proving the Overfull Conjecture, due to the results presented by Niessen (1994, 2001). In particular, the author showed the following:

THEOREM 6.2 (Niessen, 2001). *If $\Delta \geqslant n/2$, S is the set of all vertices adjacent to at most one* proper *major of G, and $v_1,\dots,v_r$ are the vertices of $G-S$ sorted in a manner that $d_{G-S}(v_1) \geqslant \cdots \geqslant d_{G-S}(v_r)$, then G is* SO *if and only if $\Delta(G-S) = \Delta(G)$ and $G[\{v_1,\dots,v_i\}]$ is overfull for some odd i satisfying either $i = r$, or $i > |V(G)| - \Delta$ and $d_{G-S}(v_i) \geqslant d_{G-S}(v_{i+1}) + 2$.*

Therefore, in order to prove the Overfull Conjecture for graphs with $\Delta \geqslant n/2$, it suffices to prove:

- first, that if u is a vertex adjacent to at most one proper major of a graph G, and if $G-u$ is *Class 1*, then G is *Class 1* (this also implies Conjecture 6.1);
- second, that if $\Delta(G) \geqslant n/2$, S is the set of all vertices adjacent to at most one proper major of G, $v_1,\dots,v_r$ are the vertices of $G-S$ sorted in a manner that $d_{G-S}(v_1) \geqslant \cdots \geqslant d_{G-S}(v_r)$, and i is an integer in $\{1,\dots,r-1\}$ such that $G[\{v_1,\dots,v_{i+1}\}]$ is not overfull and $G[\{v_1,\dots,v_i\}]$ has a $\Delta(G)$-edge-colouring, then $G[\{v_1,\dots,v_{i+1}\}]$ also has a $\Delta(G)$-edge-colouring.

6.2 Final remarks on edge-colouring complementary prisms

Table 6.2 presents the state of the art on edge-colouring prisms and complementary prisms. We refer the reader to Observation 1.12 (p. 26) for the standard result on edge-colouring prisms listed in the table.

Table 6.2: State of the art then and now on edge-colouring prisms and complementary prisms. Recall that the ∅ symbol indicates that the corresponding graph class had not been approached by any work in the literature, to the best of our knowledge.

	Then		Now	
Graph class	Status	References	Status	Theorems
prisms	solved	standard		
complementary prisms	open	∅	open	
↳ which are regular	open	∅	open	
↳ which are not regular	open	∅	solved	1.13 (p. 27)

In Chapter 4 we have raised the question of which regular complementary prisms are *Class 1* and which are *Class 2*. This seems to be a very important question, since *Class 2* regular complementary prisms would serve as examples of *Class 2* non-*SO* d-regular graphs, generalising what is already known to happen if $d = 3$ (recall the discussion on snarks in Chapter 2).

Also in Chapter 4, we have discussed some properties about regular complementary prisms. Another interesting property is the following:

- If there is a *Class 1* n-vertex regular complementary prism $G\overline{G}$, then for any colour α used in any edge-colouring of $G\overline{G}$, the numbers of α-coloured edges in

G, in $\overline{G}$, and in M are each at least one, being the number of α-coloured edges in M odd.

This follows because the set of α-coloured edges is a perfect matching in $G\overline{G}$ which contains at most $(n-1)/2$ edges of G and at most this same amount of edges of $\overline{G}$. Therefore, there must be an edge in M which is coloured α. However, it cannot be the case that only edges in M are coloured α, since otherwise both G and $\overline{G}$ would be *Class 1*, when they both are actually overfull. The fact that the number of α-coloured edges in M is odd follows from Lemma 4.3 (p. 66), since if one of the colours is assigned to an even number of edges in M, then all the colours are, which cannot happen because $|M| = n$ is odd.

6.3 Final remarks on edge-colouring join graphs

As we have argued in Chapters 1 and 2, the Overfull Conjecture (Conjecture 2.18, p. 44) suggests the existence of a linear-time algorithm for determining the chromatic index of join graphs. However, no such algorithm has been found yet, despite many partial and technical results on edge-colouring join graphs being reported by several authors. Our work presents more of these works, as Table 6.3 summarises.

Table 6.3: State of the art then and now on edge-colouring join graphs. Recall that we always assume $n_1 \leqslant n_2$, without loss of generality. Also, remark that the subclasses listed may have non-empty intersection.

	Then		Now	
Graph class	Status	References	Status	Theorems
join graphs	open		open	
↳ regular join graphs	solved	[1]		
↳ graphs with a universal vertex	solved	[2]		
↳ cographs	open		open	
↳ quasi-thresholds	solved	[2]		
↳ complete multipartite graphs	solved	[3]		
↳ with $\Delta_1 > \Delta_2$	solved	[4]		
↳ with $\Delta_2 > \Delta_1$	open	[5]	open	
↳ with $\Delta_1 = \Delta_2$	open	[6,7,4,8]	open	4.9 (p. 70), 4.10 (p. 72), 4.13 (p. 73), 4.15 (p. 73), 4.18 (p. 75)

[1] De Simone and Galluccio (2007) [2] Plantholt (1981) [3] Hoffman and Rodger (1992) [4] De Simone and Mello (2006) [5] Machado and Figueiredo (2010) [6] De Simone and Galluccio (2009) [7] De Simone and Galluccio (2013) [8] Cunha Lima et al. (2015)

Besides the technical results referenced in Table 6.3, we have conjectured in Chapter 1 that a join graph with $\Delta_1 \geqslant \Delta_2$ is *Class 1* whenever $\Lambda[G_1]$ is acyclic. In Chapter 4, we have presented evidences for this conjecture and an attempt to prove it by an extended recolouring procedure, which seems to be a promising strategy.

6.4 Final remarks on edge-colouring chordal graphs and circular-arc graphs

It is widely suspected that the chromatic index of chordal graphs can be characterised by a linear-time decidable property, namely, *neighbourhood-overfullness* (recall Chapter 2). However, as for join graphs, the actual computational complexity of the problem remains open. Although chordal graphs have not been initially the subject of our concerns, one of the decomposition techniques developed while studying the other graph classes have led us to completely solve the problem for chordal graphs with $\Delta \leqslant 3$. Table 6.4 places our result in the state of the art of edge-colouring chordal graphs, along with the results which we have found for circular-arc graphs.

Table 6.4: State of the art then and now on edge-colouring circular-arc graphs and chordal graphs. Recall that indifference graphs are a subclass both of circular-arc graphs and of chordal graphs. Recall that the $\varnothing$ symbol indicates that the corresponding graph class had not been approached by any work in the literature, to the best of our knowledge.

	Then		Now	
Graph class	Status	References	Status	Theorems
circular-arc graphs	open	$\varnothing$	open	
↳ indifference graphs	open		open	
↳ with odd Δ	solved	[1]		
↳ split-indifference graphs	solved	[2]		
↳ which are twin-free	solved	[3]		
↳ with $n \equiv 0 \pmod{(\Delta+1)}$	open	$\varnothing$	solved	5.12 (p. 96)
↳ with odd Δ	open	$\varnothing$	open	
↳ with a maximal clique of size 2	open	$\varnothing$	open	
↳ with $n \not\equiv 1, \Delta \pmod{(\Delta+1)}$	open	$\varnothing$	solved	5.13 (p. 97)
chordal graphs	open	[4]	open	
↳ split-comparability graphs	solved	[5]		
↳ with $\Delta \leqslant 3$	open	$\varnothing$	solved	4.36 (p. 86)

[1] Figueiredo et al. (1997b) [2] Ortiz Z. et al. (1998) [3] Figueiredo et al. (2003) [4] Figueiredo et al. (2000) [5] Sousa Cruz et al. (2017)

As the proof of Theorem 4.36 (p. 86) goes by decomposing the graph into subgraphs of the K_4 at its articulation points (i.e. at its separating K_1's), we suspect that analogous decompositions for $\Delta \geqslant 3$ may lead to the proof that chordal graphs of odd Δ are *Class 1*.

Let $\mathscr{A}$ denote the class of the proper circular-arc graphs with odd Δ and a maximal clique of size two (recall Chapter 5). Since we know that that there are overfull graphs in $\mathscr{A}$ (Figure 5.3, p. 98) and that all *SO* graphs in $\mathscr{A}$ are overfull (Corollary 5.14, p. 97), we propose the following conjecture:

Conjecture 6.3. *A graph in $\mathscr{A}$ is* Class 2 *if and only if it is overfull.*

6.5 Final remarks on total colouring

While developing our results on edge-colouring, we have also come to some results on total colouring join graphs, cobipartite graphs, and circular-arc graphs. Being the whole set of our results on total colouring too technical to organise in a table, we refer the reader to Chapter 5 for more details.

Besides the results on total colouring restricted to these graph classes, we have also proposed a recolouring heuristic for total colouring general graphs. Concerning this heuristic, some questions arise which should be very interesting to investigate:

- In which graph classes this recolouring heuristic may lead to novel results on total colouring?

- How does this heuristic empirically behave for constructing $(\Delta + k)$-total colourings considering, for example, random graph models such as $\mathscr{G}(n,p)$ and $\mathscr{G}(n,d)$ graphs?

- In the cases wherein the procedure does not work for any possible enumeration of missing colours at the vertices of the recolouring fan, is there some way to recolour the edges in order to obtain a recolouring fan satisfying one of the halting conditions? Remark that a positive answer of this question, when $k = 2$, implies the Total Colouring Conjecture. Remark also that a counterexample would be a graph G and an edge uv of G such that, for every $(\Delta(G) + 2)$-total colouring of $G - uv$, the procedure fails for every possible enumeration of missing colours. We suspect that such a graph with such an edge does not exist.

In spite of total and edge-colouring problems being hard combinatorial problems, even for graphs classes in which other hard problems are easy, there are several interesting conjectures and open problems on the subject. One important example is the Overfull Conjecture on edge-colouring, which has been open for over 30 years even for graphs with $\Delta \geqslant n/2$ or triangle-free graphs with $\Delta > n/3$. Our work contributes towards the proof of this conjecture, bringing also other results on edge-colouring, and raising further questions which should be interesting to investigate.

Graph Edge-colouring is surely one of the infinitely many wonders on the Wonderland of Graph Theory.

References

Aaronson, S. (2017). $\mathcal{P} \stackrel{?}{=} \mathcal{NP}$. URL: `https://www.scottaaronson.com/papers/pnp.pdf`. Accessed: 14–12–2018.

Akbari, S., Cariolaro, D., Chavooshi, M., Ghanbari, M., and Zare, S. (2012). Some criteria for a graph to be Class 1. *Discrete Math.*, 312:2593–2598.

Akbari, S. and Ghanbari, M. (2012). The chromatic index of a graph whose core has maximum degree 2. *Electron. J. Combin.*, 19.

Alon, N. and Stav, U. (2008). The maximum edit distance from hereditary graph properties. *J. Comb. Theory B*, 98:672–697.

Appel, K. and Haken, W. (1977). Every planar map is four colorable. Part I: discharging. *Illinois J. Math.*, 21:429–490.

Appel, K., Haken, W., and Koch, J. (1977). Every planar map is four colorable. Part II: reducibility. *Illinois J. Math.*, 21:491–567.

Behzad, M. (1965). *Graphs and their chromatic numbers*. PhD thesis, Michigan State University.

Behzad, M., Chartrand, G., and Cooper Jr., J. K. (1967). The color numbers of complete graphs. *J. London Math. Soc.*, 42:226–228.

Beineke, L. W. and Wilson, R. J. (1973). On the edge-chromatic number of a graph. *Discrete Math.*, 5:15–20.

Berge, C. (1957). Two theorems in graph theory. *Proc. Natl. Acad. Sci. U.S.A.*, 43(9):842–844.

Berge, C. (1961). Färbung von Graphen, deren sämtliche bzw. deren ungerade Kreise starr sind. *Wiss. Z. Martin-Luther-Univ. Halle-Wittenberg Math.-Natur. Reihe*, 10.

Björklund, A., Husfeldt, T., and Kolvisto, M. (2009). Set partitioning via inclusion-exclusion. *SIAM J. Comput.*, 39(2):546–563.

Blanuša, D. (1946). Problem četiriju boja. *Glasnik. Mat. Fiz. Astr. Ser. II*, 1:31–42.

Bodlaender, H. L. (1990). Polynomial algorithms for graph isomorphism and chromatic index on partial k-trees. *J. Algorithms*, 11:631–641.

Bollobás, B. (2001). *Random graphs*. Cambridge University Press, 2$^{\text{nd}}$ edition.

Bondy, J. A. and Murty, U. S. R. (2008). *Graph Theory*. Springer.

Booth, K. S. and Lueker, G. S. (1976). Testing for the consecutive ones property, interval graphs, and graph planarity using PQ-tree algorithms. *J. Comput. Syst. Sci.*, 13:335–379.

Brooks, R. L. (1941). On colouring the nodes of a network. *Proc. Cambridge Philos. Soc., Math. Phys. Sci.*, 37:194–197.

Burke, E., de Werra, D., and Kingston, J. (2004). Appications to timetabling. In Gross, J. L. and Yellen, J., editors, *Handbook of Graph Theory*. CRC, 1st edition.

Cai, L. and Ellis, J. A. (1991). $\mathcal{NP}$-completeness of edge-colouring some restricted graphs. *Discrete Appl. Math.*, 30:15–27.

Campos, C. N., Figueiredo, C. M. H., Machado, R., and Mello, C. P. (2012). The total chromatic number of split-indifference graphs. *Discrete Math.*, 312:2690–2693.

Cariolaro, D. and Cariolaro, G. (2003). Colouring the petals of a graph. *Electron. J. Combin.*, 10.

Chetwynd, A. G. and Hilton, A. J. W. (1984a). The chromatic index of graphs of even order with many edges. *J. Graph Theory*, 8:463–470.

Chetwynd, A. G. and Hilton, A. J. W. (1984b). Partial edge-colourings of complete graphs or of graphs which are nearly complete. In *Graph theory and combinatorics: Proc. Cambridge Combinatorial Conference in honour of Paul Erdős*, pages 81–98, London. Academic Press.

Chetwynd, A. G. and Hilton, A. J. W. (1985). Regular graphs of high degree are 1-factorizable. *Proc. London Math. Soc.*, 3(50):193–206.

Chetwynd, A. G. and Hilton, A. J. W. (1986). Star multigraphs with three vertices of maximum degree. *Math. Proc. Cambridge Philos. Soc.*, 100:303–317.

Chetwynd, A. G. and Hilton, A. J. W. (1989a). 1-factorizing regular graphs of high degree: an improved bound. *Discrete Math.*, 75:103–112.

Chetwynd, A. G. and Hilton, A. J. W. (1989b). The edge-chromatic class of graphs with maximum degree at least $|V|-3$. *Ann. Discrete Math.*, 41:91–110.

Chladný, M. and Škoviera, M. (2010). Factorisation of snarks. *Electron. J. Combin.*, 17(#R32).

Chudnovsky, M., Cornuéjols, G., Liu, X., Seymour, P., and Vušković, K. (2005). Recognizing berge graphs. *Combinatorica*, 25:143–186.

Chudnovsky, M., Robertson, N., Seymour, P., and Thomas, R. (2006). The strong perfect graph theorem. *Annals of Mathematics*, 164:51–229.

Chvátal, V. (1975). On certain polytopes associated with graphs. *J. Combin. Theory Ser. B*, 18:138–154.

Corneil, D. G., Lerchs, H., and Burlingham, L. S. (1981). Complement reducible graphs. *Discrete Appl. Math.*, 3:163–174.

Corneil, D. G., Perl, Y., and Stewart, L. K. (1985). A linear recognition algorithm for cographs. *SIAM J. Comput.*, 14(4):926–934.

Corrádi, K. and Hajnal, A. (1963). On the maximal number of independent circuits in a graph. *Acta Mathematica Academiae Scientiarum Hungaricae*, 14:423–439.

Costa, V., Dantas, S., and Rautenbach, D. (2015). The burning of the snark. *Matemática Contemporânea*, 44:1–10.

Cunha Lima, A. R., Garcia, G., Zatesko, L. M., and Almeida, S. M. (2015). On the chromatic index of cographs and join graphs. In *Proc. VIII Latin-American Algorithms, Graphs, and Optimization Symposium (LAGOS '15)*, volume 50 of Electron. Notes Discrete Math., pages 433–438, Beberibe, Brazil.

Dailey, D. P. (1980). Uniqueness of colorability and colorability of planar 4-regular graphs are $\mathcal{NP}$-complete. *Discrete Math.*, 30(3):289–293.

De Simone, C. and Galluccio, A. (2007). Edge-colouring of regular graphs of large degree. *Theor. Comput. Sci.*, 389:91–99.

De Simone, C. and Galluccio, A. (2009). Edge-colouring of join of regular graphs. *J. Comb. Optim.*, 18:417–428.

De Simone, C. and Galluccio, A. (2013). Edge-colouring of joins of regular graphs II. *J. Comb. Optim.*, 25:78–90.

De Simone, C. and Mello, C. P. (2006). Edge-colouring of join graphs. *Theor. Comput. Sci.*, 355:364–370.

Descartes, B. (1948). Network-colourings. *The Mathematical Gazette*, 32:67–69.

Diestel, R. (2010). *Graph Theory*. Springer, 4$^{\text{th}}$ edition.

Erdős, P. and Wilson, R. J. (1977). On the chromatic index of almost all graphs. *J. Comb. Theory B*, 23:255–257.

Erdős, P. (1964). Problem 9. In Fieldler, M., editor, *Theory of Graphs and its Applications*, page 159. Prague.

Erdős, P., Rubin, A. L., and Taylor, H. (1979). Choosability in graphs. In *Proc. West Coast Conference on Combinatorics, Graph Theory, and Computing*, number 26 in Congr. Numer., pages 125–157.

Erlebach, T. and Jansen, K. (2001). The complexity of path coloring and call scheduling. *Theor. Comput. Sci.*, 255(1–2):33–50.

Figueiredo, C. M. H., Meidanis, J., and Mello, C. P. (1997a). Coloração em grafos. Technical report presented at Informatics Up-to-Dating Journey (JAI), 17$^{\text{th}}$ Congress of the Brazilian Computer Society (CSBC '97). URL: `https://www.cos.ufrj.br/~lfignacio/teoriadegrafos/jai97.pdf`. Accessed: 14–12–2018.

Figueiredo, C. M. H., Meidanis, J., and Mello, C. P. (1997b). On edge-colouring indifference graphs. *Theor. Comput. Sci.*, 181:91–106.

Figueiredo, C. M. H., Meidanis, J., and Mello, C. P. (1999). Total-chromatic number and chromatic index of dually chordal graphs. *Inf. Process. Lett.*, 70:147–152.

Figueiredo, C. M. H., Meidanis, J., and Mello, C. P. (2000). Local conditions for edge-coloring. *J. Combin. Math. Combin. Comput.*, 32:79–91.

Figueiredo, C. M. H., Meidanis, J., Mello, C. P., and Ortiz, C. (2003). Decompositions for the edge colouring of reduced indifference graphs. *Theor. Comput. Sci.*, 297:145–155.

Folkman, J. and Fulkerson, D. R. (1969). Edge colorings in bipartite graphs. *Comb. Math. Appl.*, pages 561–577.

Fournier, J.-C. (1977). Méthode et théorème générale de coloration des arêtes. *J. Math. Pures Appl.*, 56:437–453.

Fulkerson, D. R. and Gross, O. A. (1965). Incidence matrices and interval graphs. *Pacific J. Math.*, 15:835–855.

Galby, E., Lima, P. T., Paulusma, D., and Ries, B. (2018). Classifying k-edge colouring for h-free graphs. Submitted. URL: `https://arxiv.org/abs/1810.04379`. Accessed: 29–10–2018.

Galvin, F. (1995). The list chromatic index of a bipartite multigraph. *J. Comb. Theory B*, 63:153–158.

Gandham, S., Dawande, M., and Prakash, R. (2005). Link scheduling in sensor networks: distributed edge coloring revisited. In *Proc. 24th INFOCOM*, pages 2492–2501.

Gardner, M. (1976). Mathematical games. *Scientific American*, 4(234):126–130.

Garey, M. R., Johnson, D. S., Miller, G. L., and Papadimitriou, C. H. (1980). The complexity of coloring circular arcs and chords. *SIAM J. Algebraic Discrete Methods*, 1(2):216–227.

Gravier, S. (1996). A Hajós-like theorem for list coloring. *Discrete Math.*, 152:299–302.

Grötschel, M., Lovász, L., and Schrijver, A. (1988). *Geometric Algorithms and Combinatorial Optimization*. Springer-Verlag.

Häggkvist, R. and Janssen, J. (1997). New bounds on the list-chromatic index of the complete graph and other simple graphs. *Combin. Probab. Comput.*, 6:295–313.

Hajnal, A. and Szemerédi, E. (1970). Proof of a conjecture of P. Erdős. In *Combinatorial theory and its applications (Proc. Colloq., Balatonfüred)*, pages 601–623, Budapest. North-Holland.

Harary, F. and Palmer, E. M. (1973). *Graphical Enumeration*. Academic Press, New York.

Haynes, T. W., Henning, M. A., and Merwe, L. C. (2009). Domination and total domination in complementary prisms. *J. Comb. Optim.*, 18:23–37.

Hilton, A. J. W. (1984). Two conjectures on edge-colouring. *Discrete Math.*, 74:61–64.

Hilton, A. J. W. (1990). A total-chromatic number analogue of Plantholt's theorem. *Discrete Math.*, 79:169–175.

Hilton, A. J. W. (1991). The total chromatic number of nearly complete bipartite graphs. *J. Comb. Theory B*, 52:9–19.

Hilton, A. J. W. and Cheng, Z. (1992). The chromatic index of a graph whose core has maximum degree two. *Discrete Math.*, 101:135–147.

Hilton, A. J. W. and Hind, H. R. (1993). The total chromatic number of graphs having large maximum degree. *Discrete Math.*, 117:127–140.

Hilton, A. J. W. and Johnson, P. D. (1987). Graphs which are vertex-critical with respect to the edge-chromatic number. *Math. Proc. Cambridge Philos. Soc.*, 102:103–112.

Hind, H., Molloy, M., and Reed, B. (2000). Total colouring with $\delta + \text{poly}(\log \delta)$ colours. *SIAM J. Comput.*, 28:816–821.

Hind, H. R. (1990). An upper bound for the total chromatic number. *Graphs Combin.*, 6:153–159.

Hoffman, D. G. and Rodger, C. A. (1992). The chromatic index of complete multipartite graphs. *J. Graph Theory*, 16:159–163.

Holton, D. A. and Sheehan, J. (1992). *The Petersen Graph*. Cambridge University Press.

Holyer, I. (1981). The $\mathcal{NP}$-completeness of edge-colouring. *SIAM J. Comput.*, 10(4):718–720.

Isaacs, R. (1975). Infinite families of non-trivial trivalent graphs which are not Tait-colorable. *Amer. Math. Monthly*, 82:221–239.

Ito, H. and Yokoyama, M. (1998). Linear time algorithms for graph search and connectivity determination on complement graphs. *Inf. Process. Lett.*, 66:209–213.

Jaffe, A. M. (2000). The Millenium Grand Challenge in Mathematics. *Notices of the AMS*, 53(6):652–660.

Janssen, J. C. M. (1993). The Dinitz problem solved for rectangles. *Bull. Amer. Math. Soc.*, 29:243–249.

Januario, T., Urrutia, S., Ribeiro, C. C., and Werra, D. (2016). Edge coloring: A natural model for sports scheduling. *Eur. J. Oper. Res*, 254:1–8.

Jensen, T. R. and Toft, B. (1994). *Graph Coloring Problems*. Wiley-Interscience.

Johnson, D. S. (1985). The $\mathcal{NP}$-completeness column: an ongoing guide. *J. Algorithms*, 6:434–451.

Karp, R. M. (1972). Reducibility among combinatorial problems. In Miller, R. E. and Thatcher, J. W., editors, *Complexity of Computer Computations*, pages 85–103. Plenum.

Kempe, A. B. (1886). A memoir on the theory of mathematical form. *Philos. Trans. Royal Soc.*, 177:1–70.

Kőnig, D. (1916). Graphok és alkalmazásuk a determinánsok és a halmazok elméletére. *Math. Természettudományi Értesito*, 34:104–119.

Koreas, D. P. (1997). The $\mathcal{NP}$-completeness of chromatic index in triangle free graphs with maximum vertex of degree 3. *J. Appl. Math. Comput.*, 93:13–17.

Král, D., Kratochvíl, J., Tuza, Z., and Woeginger, G. J. (2001). Complexity of coloring graphs without forbidden induced subgraph. In *Proc. 27th International Workshop on Graph-Theoretic Concepts in Computer Science (WG '01)*, volume 2204 of Lecture Notes Comput. Sci., pages 254–262.

Kubale, M. (2004). *Graph Colorings*. American Mathematical Soc.

Lekkerkerker, C. G. and Boland, J. C. (1962). Representation of a finite graph by a set of intervals on the real line. *Fund. Math.*, 51:45–64.

Leven, D. and Galil, Z. (1983). $\mathcal{NP}$-completeness of finding the chromatic index of regular graphs. *J. Algorithms*, 4:35–44.

Looges, P. J. and Olanu, S. (1993). Optimal greedy algorithms for indifference graphs. *Comput. Math. Appl.*, 25:15–25.

Lovász, L. (1972). Normal hypergraphs and the perfect graph conjecture. *Discrete Math.*, 2:253–267.

Machado, R. C. S., de Figueiredo, C. M. H., and Trotignon, N. (2013). Edge-colouring and total-colouring chordless graphs. *Discrete Math.*, 313:1547–1552.

Machado, R. C. S. and Figueiredo, C. M. H. (2010). Decompositions for edge-coloring join graphs and cobipartite graphs. *Discrete Appl. Math.*, 158:1336–1342.

Machado, R. C. S., Figueiredo, C. M. H., and Vušković, K. (2010). Chromatic index of graphs with no cycle with a unique chord. *Theor. Comput. Sci.*, 411:1221–1234.

McConnell, R. M. (2003). Linear-time recognition of circular-arc graphs. 37:93–147.

McDiarmid, C. J. H. and Sánchez-Arroyo, A. (1994). Total colouring regular bipartite graphs is $\mathcal{NP}$-hard. *Discrete Math.*, 124:155–162.

McKay, B. D. and Wormald, N. C. (1990). Asymptotic enumeration by degree sequence of graphs of high degree. *Eur. J. Combin.*, 11:565–580.

Meidanis, J. (1998). Edge coloring of cycle powers is easy. Unpublished manuscript. URL: http://www.ic.unicamp.br/~meidanis/research/edge/cpowers.ps. Accessed: 13–12–2018.

Menger, K. (1927). Zur allgemeinen kurventheorie. *Fund. Math.*, 10:96–115.

Meringer, M. (1997). Connected 4-regular graphs on 9 vertices. URL: http://www.mathe2.uni-bayreuth.de/markus/REGGRAPHS/09_4_3.html. Accessed: 29–10–2018.

Möhring, R. H. (1985). Algorithmic aspects of the substitution decomposition in optimization over relations, set systems and Boolean functions. *Ann. Oper. Res.*, 4:195–225.

Molloy, M. and Reed, B. (1998). A bound on the total chromatic number. *Combinatorica*, 18:241–280.

Naserasr, R. and Škrekovski, R. (2003). The Petersen graph is not 3-edge-colourable — a new proof. *Discrete Math.*, 268:325–326.

Niessen, T. (1994). How to find overfull subgraphs in graphs with large maximum degree. *Discrete Appl. Math.*, 51:117–125.

Niessen, T. (2001). How to find overfull subgraphs in graphs with large maximum degree, II. *Electron. J. Combin.*, 8.

Niessen, T. and Volkmann, L. (1990). Class 1 conditions depending on the minimum degree and the number of vertices of maximum degree. *J. Graph Theory*, 14:225–246.

Orlin, J. B., Bonuccelli, M. A., and Boveti, D. P. (1981). An $O(n^2)$ algorithm for coloring proper circular arc graphs. *SIAM J. Alg. Discrete Meth.*, 2(2):88–93.

Ortiz Z., C., Maculan, N., and Szwarcfiter, J. L. (1998). Characterizing and edge-colouring split-indifference graphs. *Discrete Appl. Math.*, 82:209–217.

Padberg, M. W. and Rao, M. R. (1982). Odd minimum cut-sets and b-matching. *Math. Oper. Res.*, 7:67–80.

Papadimitriou, C. H. (1994). *Computational Complexity*. Addison-Wesley, 1st edition.

Perkovic, L. and Reed, B. (1997). Edge coloring regular graphs of high degree. *Discrete Appl. Math.*, 165/166:567–578.

Petersen, J. (1898). Sur le théorème de Tait. *L'Intermédiaire des Mathématiciens*, 5:225–227.

Plantholt, M. J. (1981). The chromatic index of graphs with a spanning star. *J. Graph Theory*, 5:45–53.

Plantholt, M. J. (1983). On the chromatic index of graphs with large maximum degree. *Discrete Math.*, 47:91–96.

Roberts, F. S. (1969). Indifference graphs. In *Proc. 2nd Ann Arbor Graph Theory Conference*, pages 139–146, Ann Arbor, USA.

Robertson, N., Sanders, D. P., Seymour, P., and Thomas, R. (1996). Efficiently four-coloring planar graphs. In *Proc. 28th Symposium on Theory of Computing*, pages 571–575. ACM.

Rose, D., Lueker, G., and Tarjan, R. E. (1976). Algorithmic aspects of vertex elimination on graphs. *SIAM J. Comput.*, 5(2):266–283.

Sánchez-Arroyo, A. (1989). Determining the total colouring number is $\mathcal{NP}$-hard. *Discrete Math.*, 78:315–319.

Sanders, D. P. and Zhao, Y. (2001). Planar graphs of maximum degree 7 are Class I. *J. Comb. Theory B*, 83:201–212.

Schauz, U. (2014). Proof of the list edge coloring conjecture for complete graphs of prime degree. *Electron. J. Combin.*, 21.

Sipser, M. (1992). The history and status of the $\mathcal{P}$ versus $\mathcal{NP}$ problem. In *Proc. 24th Annual ACM Symposium on the Theory of Computing*, pages 603–618.

Skiena, S. S. (2008). *The Algorithm Design Manual*. Springer-Verlag, 2nd edition.

Skupień, Z. (2007). Exponentially many hypohamiltonian snarks. *Electron. Notes Discrete Math.*, 28:417–424.

Sousa Cruz, J. B., Silva, C. N., and Almeida, S. M. (2017). The Overfull Conjecture on split-comparability graphs. Submitted. URL: `https://arxiv.org/pdf/1710.03524.pdf`. Accessed: 29–10–2018.

Stiebitz, M., Scheide, D., Toft, B., and Favrholdt, L. M. (2012). *Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture*. Wiley.

Swart, E. R. (1980). The philosophical implications of the four-color problem. *Amer. Math. Monthly*, 87(9):697–702.

Tait, P. G. (1878–1880). On the colouring of maps. *Proc. Roy. Soc. Edinburgh Sect. A*, 10:501–503,729.

Vizing, V. G. (1964). On an estimate of the chromatic class of a p-graph (in Russian). *Diskret. Analiz.*, 3:25–30.

Vizing, V. G. (1965). Critical graphs with a given chromatic class (in Russian). *Diskret. Analiz.*, 5:9–17.

Vizing, V. G. (1968). Some unsolved problems in graph theory. *Russian Math. Surveys*, 23:125–141.

Vizing, V. G. (1976). Vertex colorings with given colors (in Russian). *Metody Diskret. Analiz.*, 29:3–10.

Williamson, D. P., Hall, L. A., Hoogeveen, J. A., Hurkens, C. A. J., Lenstra, J. K., Sevast'janov, S. V., and Shmoys, D. B. (1997). Short shop schedules. *Oper. Res.*, 45(2):288–294.

Appendix A: List of Publications

Earlier versions of the recolouring procedure described in Chapter 3, as well as some of the results yielded on edge-colouring general graphs and triangle-free graphs with bounded local degree sums, have been published in:

- Zatesko, L. M., Carmo, R., and Guedes, A. L. P. (2017). Edge-colouring of triangle-free graphs with no proper majors. In *Proc. 37th Congress of the Brazilian Computer Society (CSBC '17/II ETC)*, pages 71–74, São Paulo.
- Zorzi, A. and Zatesko, L. M. (2018). On the chromatic index of join graphs and triangle-free graphs with large maximum degree. *Discrete Appl. Math.*, 245:183–189.

The former work also contains some of the results on edge-colouring join graphs presented in Chapter 4.

The recolouring procedure discussed in Chapter 4 in the context of our main conjecture on join graphs (Conjecture 1.11) has been presented in:

- Zatesko, L. M., Carmo, R., and Guedes, A. L. P. (2017). On a conjecture on edge-colouring join graphs. In *Proc. 2nd Workshop de Pesquisa em Computação dos Campos Gerais (WPCCG '17)*, pages 69–72, Ponta Grossa, Brazil.

The results on total colouring join graphs and cobipartite graphs in Chapter 5 have been presented in:

- Zatesko, L. M., Carmo, R., and Guedes, A. L. P. (2018). Upper bounds for the total chromatic number of join graphs and cobipartite graphs. In *Proc. 7th International Conference on Operations Research and Enterprise Systems (ICORES '18)*, pages 247–253, Funchal.

The results on total and edge-colouring circular-arc graphs, also in Chapter 5, have been presented in:

- Bernardi, J. P. W., Almeida, S. M., and Zatesko, L. M. (2018). On total and edge-colouring of proper circular-arc graphs. In *Proc. 38th Congress of the Brazilian Computer Society (CSBC '18/III ETC)*, pages 73–76, Natal.

The decomposition technique in Chapter 4 which yields our result on edge-colouring chordal graphs with $\Delta \leqslant 3$ shall be submitted to Matemática Contemporânea and has been presented in:

- Bernardi, J. P. W., Almeida, S. M., and Zatesko, L. M. (2018). A decomposition for edge-colouring. In *Proc. VIII Latin American Workshop on Cliques in Graphs*, page 35, Rio de Janeiro.

Also, our recolouring heuristic for total colouring (which is being implemented and whose empirical results shall appear in a submission also to Matemática Contemporânea) has been presented in:

- Zatesko, L. M., Carmo, R., and Guedes, A. L. P. (2018). A recolouring procedure for total colouring. In *Proc. VIII Latin American Workshop on Cliques in Graphs*, page 31, Rio de Janeiro.

The current and stronger version of the recolouring procedure of Chapter 3, as well as the results presented in Chapters 3 and 4 which cannot be found in the aforementioned works, have been submitted to Discrete Applied Mathematics:

- Zatesko, L. M., Zorzi, A., Carmo, R., and Guedes, A. L. P. (2018). Edge-colouring graphs with bounded local degree sums. Submitted to Discrete Appl. Math.

Our result on complementary prisms, and further results on circular-arc graphs, have been submitted to the *X Latin and American Algorithms, Graphs and Optimization Symposium* (LAGOS '19):

- Zatesko, L. M., Carmo, R., and Guedes, A. L. P. (2018). The chromatic index of prisms and non-regular complementary prisms. Submitted to the *X Latin and American Algorithms, Graphs and Optimization Symposium* (LAGOS '19).

- Bernardi, J. P. W., Silva, M. V. G., Guedes, A. L. P., and Zatesko, L. M. (2018). The chromatic index of proper circular arc graphs of odd maximum degree which are chordal. Submitted to the *X Latin and American Algorithms, Graphs and Optimization Symposium* (LAGOS '19).

Index